Part I

Basic Definitions and Examples

Outline

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Contents

1 Fundamentals

Fundamental Notation

- L is a locally compact non-archimedean field of characteristic zero. It serves as our ground field for manifolds and groups.
- $\omega : L \to \mathbf{Z}$ is the normalized valuation with $\omega(\pi) = 1$ for a uniformizing parameter π of the ring of integers o_L of L.
- K is a complete field containing L. It serves as our coefficient field. In general we assume that K is spherically complete. At times we may need more restrictive hypotheses (discretely valued or locally compact.)

Locally convex topological spaces

Let V be a topological vector space over K. The following notions arise in our lectures:

- Locally convex vector space
- Banach space
- Fréchet space
- locally convex limit topologies
- weak and strong duals
- reflexivity
- barrelledness
- compact maps
- tensor product topologies

If K is locally compact, a map $f : U \to V$ of Banach spaces over K is compact provided that the closure of the image under f of the unit ball of U in V is compact.

Spaces of compact type

A topological vector space V over K is *of compact type* if there is a sequence

$$U_1 \to U_2 \to \cdots \to U_n \to \cdots$$

of K Banach spaces linked by injective, compact linear maps such that V is isomorphic to $\varinjlim U_i$.

Such spaces have the following properties:

1. They are Hausdorff, complete and reflexive.
2. The category of vector spaces of compact type is closed under passage to closed subspaces and quotients by closed subspaces.
3. $V = \varinjlim U_i$ is of compact type if and only if the strong dual $V'_b = \operatorname{proj}\lim (U_i)'_b$ is a nuclear Fréchet space.

2 Locally analytic functions and manifolds

Analytic Manifolds

Dat's lectures introduced p-adic Lie groups and (in passing) p-adic manifolds. In his talk the focus was on $\mathbf{Q}_p$-analytic manifolds. More generally one has L-analytic manifolds having a covering by a compatible collection of analytic charts (M, ϕ)

$$\phi : M \to B_r \subseteq L^d$$

where ϕ is a homeomorphism and B_r is a closed polydisk in L^d of radius $r \in |L^*|$. (One could just take $B_r = o_L^d$ for simplicity)

Two charts (M_i, ϕ_i) and (M_j, ϕ_j) are compatible if the map

$$\phi_i \circ \phi_j^{-1} : B_{r(j)} \to B_{r(i)}$$

is locally L-analytic – meaning given by a collection of convergent power series where the variables are L-valued.

L-Analytic functions

- A function $f : B_r \to K$ is locally L-analytic if there is a finite covering of B_r by polydisks $B_{r'}$ such that the restriction of f to each $B_{r'}$ is given by a convergent power series in d-variables with coefficients in K.
- A function $f : M \to K$ is called locally L-analytic if $f \circ \phi_i^{-1}$ is locally L-analytic on B_r for any chart (M, ϕ).

We write $C^{an}(M, K)$ for the space of K-valued locally L-analytic functions on M.

The space of locally analytic functions is contained in the space $C(M, K)$ of continuous functions on M.

Vector valued functions

The study of representation theory requires L-analytic functions taking values in topological vector spaces.

Let U be a K-Banach space with norm $\|\cdot\|_U$.

Assume that $\phi_i : M_i \to L^d$ identifies M_i with the polydisc of radius $r \in |L^*|$ centered at zero in L^d.

The Banach space of U-valued L-analytic functions on M_i is

$$\mathcal{A}_K(M_i, \phi_i, U) = \{ \sum_{I=(i_1,\ldots,i_d)} a_I \mathbf{x}^I : a_I \in U\}$$

such that

$$\|a_I\|_U \; r^{|I|} \to 0 \text{ as } |I| = \sum_{j=1}^{d} i_j \to \infty.$$

Vector valued functions continued

Let V be a general Hausdorff locally convex K-vector space.

Given a covering $\mathcal{M} := (M_i, \phi_i)_{i\in I}$, and a family of injections $\mathcal{U} = \{U_i \hookrightarrow V\}_I$ of Banach spaces into V indexed by $i \in I$, set:

$$C^{an}(\mathcal{M}, \mathcal{U}) := \prod_{i\in I} \mathcal{A}_K(M_i, \phi_i, U_i).$$

The space $C^{an}(M, V)$ of *V-valued analytic functions* is the direct limit over a family $\{\mathcal{M}_j, \mathcal{U}_j\}_j$ of finer coverings and enlarged Banach spaces:

$$C^{an}(M, V) := \varinjlim C^{an}(\mathcal{M}_j, \mathcal{U}_j).$$

Vector valued functions continued

Theorem 1. *Suppose that M is compact and V is of compact type. Then $C^{an}(M, V)$ is of compact type.*

Proof. Consider the simple case when $V = K$. View o_L as an L-analytic manifold and consider πo_L inside o_L. We have

$$\begin{aligned} \mathcal{A}(o_L, K) &= \{\sum_{i=0}^{\infty} a_i x^i : a_i \in K \text{ and } \omega(a_i) \to \infty \text{ as } i \to \infty\} \\ \mathcal{A}(\pi o_L, K) &= \{\sum_{i=0}^{\infty} a_i x^i : a_i \in K \text{ and } \omega(a_i) - i \to \infty \text{ as } i \to \infty\} \end{aligned}$$

Restriction of functions from o_L to πo_L amounts to the inclusion $\mathcal{A}(o_L, K) \hookrightarrow \mathcal{A}(\pi o_L, K)$. The unit ball of $\mathcal{A}(o_L, K)$ has closure equal to the compact set of series in $\mathcal{A}(\pi o_L, K)$ having integral coefficients. $\square$

Remarks on Analytic Functions

- If V is Hausdorff, and if $M = \bigcup M_i$ is a partition of M into pairwise disjoint open subsets, then

$$C^{an}(M, V) = \prod_i C^{an}(M_i, V).$$

- The space $C^\infty(M, V)$ of locally constant V-valued functions is a closed subspace of $C^{an}(M, V)$. The elements of $C^\infty(M, V)$ are the *smooth* functions of Langlands theory.
- If M is compact and V is Hausdorff, then $C^{an}(M, V)$ injects into $C^{an}(M, K)\hat{\otimes}_\pi V$; this is a topological isomorphism if V is of compact type.

3 Locally Analytic Distributions

Distributions

If M is a locally L-analytic manifold and V a locally convex vector space, the *V-valued locally analytic distributions* are the elements of the space:

$$D(M, V) = C^{an}(M, V)'_b$$

When V is of compact type, then by reflexivity $D(M, V)$ is a Fréchet space and

$$C^{an}(M, V) = D(M, V)'_b.$$

In particular this holds when $V = K$ and K/L is finite.

The Dirac distributions δ_x, for $x \in M$, defined by $\delta_x(f) = f(x)$ are elements of $D(M, K)$.

Integration against vector valued distributions

Theorem 2. *If V is a locally convex vector space such that there exists a countable family of continuous injections $f_i : U_i \hookrightarrow V$, where each U_i is a Banach space, and the images of the U_i cover V, then*

$$\begin{array}{rcl} I^{-1} : \mathcal{L}(D(M, K), V) & \xrightarrow{\cong} & C^{an}(M, V) \\ A & \longmapsto & [x \longmapsto A(\delta_x)] \end{array}$$

is a well defined K-linear isomorphism. When M is compact and V is of compact type, then this is a topological isomorphism (for the strong topology on $\mathcal{L}(D(M, K), V)$.)

Integration continued

The basic theorem follows from the following considerations:

- When V is a Banach space and M is an open ball, one shows directly that $C^{an}(M,K)\hat{\otimes}V = \mathcal{L}(D(M,K),V)$.
- When V is as in the Theorem, then $V = \bigcup_i f_i(U_i)$ and the open mapping theorem implies that a linear map from the Fréchet space $D(M,K)$ into V actually lands in one of the $f_i(U_i)$. Thus both $C^{an}(M,V)$ and $\mathcal{L}(D(M,K),V)$ are compatible direct limits over the family of U_i.
- When M is more general, one uses the compatibility of the isomorphism with products.

Integration continued

The inverse map

$$I : C^{an}(M,V) \to \mathcal{L}(D(M,K),V)$$

is "integration" against a locally analytic distribution. In particular

$$I(f)(\delta_x) = f(x).$$

Distribution algebras

Suppose now that our locally L-analytic manifold is an L-analytic group G.

In this situation the space of locally analytic distributions becomes a ring.

Theorem 3. *$D(G,K)$ is a ring with a separately continuous multiplication and identity element given by the Dirac distribution δ_1 at the identity of G. When G is compact, $D(G,K)$ is a Fréchet algebra, meaning in particular that the multiplication map is a continuous bilinear map.*

Ring structure on distributions

The topological aspects of this theorem are somewhat subtle. In general,

$$D(G \times G,K) = D(G,K)\hat{\otimes}_\iota D(G,K)$$

The diagonal embedding $G \to G \times G$ induces a map

$$D(G,K) \to D(G \times G,K) \to D(G,K)\hat{\otimes}_\iota D(G,K).$$

and the composition of these maps yields the multiplication.

In the compact case, $D(G,K)$ is Fréchet and the inductive and projective tensor product topologies coincide. This implies continuity of the ring structure.

Relation to other rings

The continuous injection of $C^{an}(G,K) \hookrightarrow C(G,K)$ has dense image and yields an injection of the continuous distribution algebra $D^c(G,K)$ into $D(G,K)$. (Here, when G is compact, the ring $D^c(G,K)$ is the Iwasawa algebra discussed by Dat).

The closed embedding of $C^\infty(G,K) \to C^{an}(G,K)$ yields a surjective map

$$D(G,K) \to D^\infty(G,K)$$

onto the ring of "smooth distributions."

The ring $D^\infty(G,K)$ consists of all "p-adic distributions" in the old-fashioned sense, while the ring $D^c(G,K)$ consists of all "bounded distributions" in that sense.

More on smooth distributions

The Lie algebra $\mathfrak{g}$ of G acts on $C^{an}(G,K)$ via continuous endomorphisms as differential operators. In particular for $\mathfrak{z} \in \mathfrak{g}$ and $f \in C^{an}(G,K)$, we have

$$(\mathfrak{z}f)(g) = \frac{d}{dt} f(\exp(-t\mathfrak{z})g)|_{t=0}$$

Evaluation at the identity yields a map $\mathfrak{g} \hookrightarrow D(G,K)$, and by the universal property a map

$$U(\mathfrak{g}) \otimes K \to D(G,K).$$

This map is an injection. Since a locally analytic function is locally constant exactly when it is killed by $\mathfrak{g}$, we have

$$D^\infty(G,K) = D(G,K)/I(\mathfrak{g})$$

where $I(\mathfrak{g})$ is the two-sided ideal generated by $\mathfrak{g}$.

4 A prototypical example

Prototypical example: $G = \mathbf{Z}_p$

We consider the explicit case $G = \mathbf{Z}_p$.

Everyone should know the following classic theorem of Mahler:

Theorem 4. *Let f be a continuous function on $\mathbf{Z}_p$. Then f has a unique representation*

$$f = \sum_{n=0}^{\infty} T_n(f) \binom{x}{n}$$

where the coefficients $T_n(f) \in K$ go to zero as $n \to \infty$. Conversely any such series converges uniformly to a continuous function. The norm $\|f\| = \max_n |T_n(f)|$.

As a vector space, the dual to $C(\mathbf{Z}_p, K)$ is thus seen to be the space of bounded sequences of elements of K, with the sup-norm.

Fourier Theory

A distribution is determined by its values on the characters of finite order

$$\chi_\zeta(x) = \zeta^x \text{ for } \zeta \text{ a } p\text{-power root of } 1.$$

Then

$$T_n(\zeta^x) = T_n(((\zeta-1)+1)^x) = T_n(\sum \binom{x}{i}(\zeta-1)^i) = (\zeta-1)^n$$

and

$$T_n * T_m(\zeta^x) = T_n^{(x)} T_m^{(y)}(\zeta^{x+y}) =$$
$$T_n(\zeta^x)T_m(\zeta^y) = (\zeta-1)^{m+n} = T_{n+m}(\zeta^x).$$

It follows inductively that $T_n = T_1^n$ and so, writing $T_1 = T$, $D^c(\mathbf{Z}_p, K)$ is the space of power series in T over K with bounded coefficients.

Analytic distributions on $\mathbf{Z}_p$

Theorem 5. *An element $f \in C(\mathbf{Z}_p, K)$ is locally analytic if there is an $r > 1$ such that*

$$\lim_{n\to\infty} |T_n(f)|r^n \to 0$$

as $n \to \infty$. The dual space $D(\mathbf{Z}_p, K)$ is given by all series

$$\mu = \sum_{n=0}^{\infty} b_n T_n$$

such that, for all $r < 1$ in $p^{\mathbf{Q}}$ we have $|b_n|r^n \to 0$. The Fréchet topology on $D(\mathbf{Z}_p, K)$ is defined by the family of seminorms q_r for $r \in p^{\mathbf{Q}}$, $r < 1$, with

$$q_r(f) = \max_n |b_n|r^n.$$

Analytic distributions on $\mathbf{Z}_p$ cont'd

For the ring structure one must consider *all* locally analytic characters

$$\chi_z = (1+z)^x$$

where z is any element of K with $|z| < 1$.

These are dense in $C^{an}(\mathbf{Z}_p, K)$ and a similar computation shows again that $T_n = T^n$.

Let $\mathbf{B}$ be the open unit disk around the origin viewed as a rigid variety.

$D(\mathbf{Z}_p, K)$ is thus isomorphic to the ring $\mathcal{O}(\mathbf{B})$ of all power series over K that converging on the open unit disk via $\lambda \mapsto F_\lambda(T)$ where $F_\lambda(z) = \lambda((1+z)^x)$.

Analytic distributions on $\mathbf{Z}_p$ cont'd

The ring $D^c(\mathbf{Z}_p, K) \subset D(\mathbf{Z}_p, K)$ is the subring of bounded functions on $\mathbf{B}$.

The formula

$$\frac{d}{dx}(1+T)^x = (1+T)^x \log(1+T)$$

implies that the differential operator $\frac{d}{dx}$ corresponds to the power series

$$\log(1+T) \in \mathcal{O}(\mathbf{B}).$$

The universal enveloping algebra $U(\mathfrak{g})$ for $\mathbf{Z}_p$ thus imbeds into $D(\mathbf{Z}_p, K)$ as the ring of polynomials in $\log(1+T)$.

5 L-analyticity and $\mathbf{Q}_p$-analyticity

L-analytic and $\mathbf{Q}_p$-analytic functions

If G is a locally L-analytic group with space $C^{an}(G,K)$ of L-analytic functions, one may view G at the same time as a $\mathbf{Q}_p$-analytic group (or E-analytic group for $E \subset L$).

The relationship between the L-analytic theory and the $\mathbf{Q}_p$-analytic theory is like the relationship between real and complex analysis.

Without introducing a technical definition, we will refer to the locally $\mathbf{Q}_p$-analytic group associated to G by G_0. Essentially we view the d-dimensional charts defining G as locally isomorphic to an open ball in L^d as giving instead $d[L:\mathbf{Q}_p]$-dimensional charts giving open balls in L^d viewed as $\mathbf{Q}_p^{d[L:\mathbf{Q}_p]}$.

It is easy to see that a locally L-analytic function of d-variables is a locally $\mathbf{Q}_p$-analytic function of $d[L:\mathbf{Q}_p]$-variables.

L-analytic and $\mathbf{Q}_p$-analytic functions cont'd

Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{g}_0$ be the restriction of scalars of $\mathfrak{g}$ to $\mathbf{Q}_p$, giving the Lie algebra of G_0.

Define $\mathfrak{g}^0$ as the kernel of the map

$$0 \to \mathfrak{g}^0 \to L \otimes_{\mathbf{Q}_p} \mathfrak{g}_0 \to \mathfrak{g} \to 0.$$

Theorem 6. *The space $C^{an}(G,K)$ of locally L-analytic functions is topologically isomorphic to the closed subspace $C^{an}(G_0,K)^{\mathfrak{g}^0}$ of $\mathbf{Q}_p$-analytic functions annihilated by $\mathfrak{g}^0$.*

Omitting topological complications, the annihilation condition amounts to "Cauchy-Riemann" equations that locally cut out the L-analytic functions inside the $\mathbf{Q}_p$-analytic ones.

A second fundamental example

Let $G = o_L$ be the additive group of the ring of integers in the field L. The group $G_0 = \mathbf{Z}_p^d$ where $d = [L : \mathbf{Q}_p]$.

Let $e_1, \ldots, e_d$ be a basis for o_L over $\mathbf{Z}_p$. If z is the L-coordinate on o_L and $x_1, \ldots, x_d$ are the $\mathbf{Q}_p$-coordinates, so that

$$z = x_1 d_1 + \cdots + x_d e_d$$

the chain rule gives the formula

$$\frac{\partial f}{\partial x_i} = e_i \frac{\partial f}{\partial z}.$$

The kernel $\mathfrak{g}^0$ is then generated by the C-R equations, for $i, j = 1, \ldots, d$:

$$e_j \frac{\partial}{\partial x_i} - e_i \frac{\partial}{\partial x_j}.$$

L-analytic characters for o_L

Applying the Fourier theory approach used to determine $D(\mathbf{Z}_p, K)$, we first identify a $\mathbf{Q}_p$-analytic character

$$x = \sum_{i=1}^{d} a_i e_i \mapsto (1 + t_1)^{a_1} \cdots (1 + t_d)^{a_d}$$

with the point $(t_1, \ldots, t_d)$ of $\mathbf{B}^d$.

Such a character is L-analytic provided that it satisfies the C-R equations, meaning that

$$e_j \log(1 + t_i) - e_i \log(1 + t_j) = 0 \quad i, j = 1, \ldots, d$$

These equations cut out a closed rigid subvariety $\hat{o}_L$ in $\mathbf{B}^d$ parameterizing L-analytic characters. The ring of rigid functions $\mathcal{O}(\hat{o}_L)$ is the quotient of $\mathcal{O}(\mathbf{B})$ by the ideal generated by these equations.

The ring $D(o_L, K)$

The "fourier transform" sending a distribution λ to the function

$$F_\lambda(t_1, \ldots, t_d) = \lambda(\kappa_{(t_1, \ldots, t_d)})$$

when $(t_1, \ldots, t_d) \in \hat{o}_L$ and $\kappa_{(t_1, \ldots, t_d)}$ is the corresponding L-analytic character, is a topological isomorphism

$$D(o_L, K) \to \mathcal{O}(\hat{o}_L).$$

Remarks on commutative distribution algebras

The examples of $\mathbf{Z}_p$ and o_L provide clues to the general structure theory of distribution algebras $D(G,K)$. In each case, the distribution algebras are the *global functions on a smooth (one-dimensional) rigid Stein variety.*

Such rings have good properties in general. For example, finitely generated ideals in such rings are automatically closed. Furthermore, one may pass back and forth between coherent sheaves on the underlying variety and certain classes of modules for the ring.

Later, these observations motivate the approach to general distribution algebras.

6 Definition of locally analytic representation

Locally analytic representations

Definition 7. A representation of G on a K-vector space V is *locally analytic* if V is a barrelled, Hausdorff, locally convex vector space, the G action is continuous, and, for each $v \in V$, the orbit map $g \mapsto gv$ belongs to $C^{an}(G,V)$.

Locally analytic representations cont'd

For locally analytic representations of G:

1. The G action extends to a separately continuous action of $D(G,K)$ on V. This action is defined by $\lambda * v = I(\rho_v)(\lambda)$ where I is the integration map introduced earlier.

2. Suppose G is compact. Then the functor $V \mapsto V_b'$ taking V to its strong dual induces an anti-equivalence of categories between *locally analytic representations on vector spaces of compact type over K with continuous G-equivariant linear maps* and *continuous $D(G,K)$ modules on nuclear K-Fréchet spaces with continuous $D(G,K)$-module maps.*

3. In the non-compact case one must work with separately continuous module structures, but a similar equivalence of categories exists.

Key examples of locally analytic representations

1. Smooth representations of G, where V has the finest locally convex topology. (Orbit maps are locally constant.)

2. Finite dimensional algebraic representations of G. (Orbit maps are polynomials.)

3. Locally algebraic representations – essentially tensor products of the first two kinds. (Orbit maps are locally polynomial.)

4. Locally analytic principal series: $\chi : P \to K^*$ a locally analytic character of the Borel subgroup of $G = \mathrm{GL}_n(L)$ trivial on unipotents, and

$$\mathrm{Ind}_P^G(\chi) = \{f \in C^{an}(G,K) : f(gp) = \chi^{-1}(p)f(g) \text{ for } p \in P\}$$

5. The strong dual of spaces of differential forms $\Omega^i_{\mathcal{X}}$ on Drinfeld's upper half space $\mathcal{X}$; or more generally global sections of equivariant holomorphic vector bundles on $\mathcal{X}$.

7 Fréchet-Stein Algebras and Admissibility

Remarks on the problem of admissibility

In general, the category of locally analytic representations is too complicated to admit a reasonable approach to classification.

Finding a reasonable theory requires identifying a good subcategory that:

- is abelian;
- minimizes topology in favor of algebra – so that, for example, only strict maps are allowed and strictness follows from algebraic considerations;
- contains the important examples at the end of the last lecture.

This problem arises even in the theory of continuous representations on Banach spaces and leads to the notion of continuous admissibility discussed by Dat.

As in the continuous case, we turn to algebraic properties of the ring $D(G,K)$ and its modules to find a good class of representations.

Motivating ideas

The underlying principle is that, when G is a compact p-adic Lie group, the ring $D(G,K)$ is a non-commutative version of the ring of functions on an open p-adic polydisk.

Such a polydisk is an example of a (rigid analytic) Stein space. The ring of functions on such a space is a *projective limit of Banach algebras* where *the transition maps are flat.* Further:

1. (Theorem A) The stalks of coherent sheaves are always generated by global sections.
2. (Theorem B) Stein spaces have trivial sheaf cohomology (so that one may pass from coherent sheaves to modules over $\mathcal{O}$ and back without obstructions). Since coherent sheaves form an abelian category, so do the associated modules of global sections.

These properties, suitably reinterpreted, play a crucial role in the definition of local-analytic admissibility.

8 Fréchet-Stein Algebras and Coadmissible Modules

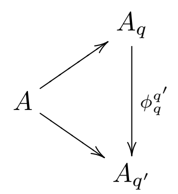
Fréchet-Stein Algebras

Let A be a K-Fréchet algebra.

Let A_q be the Banach algebra obtained as the completion of $A/\{a \in A : q(a) = 0\}$ in the quotient norm.

A_q comes with a natural continuous linear map $A \to A_q$ with dense image.

For any two continuous seminorms $q' \leq q$ the identity on A extends to a norm decreasing, linear map $\phi_q^{q'} : A_q \longrightarrow A_{q'}$ with dense image such that the diagram



commutes.

Fréchet-Stein Algebras cont'd

For any sequence $q_1 \leq q_2 \leq \ldots \leq q_n \leq \ldots$ of seminorms on A which define the Fréchet topology (such a sequence always exists), the map

$$A \xrightarrow{\cong} \varprojlim_{n \in \mathbf{N}} A_{q_n}$$

is an isomorphism of locally convex K-vector spaces.

A continuous seminorm q on A is an algebra seminorm if the multiplication on A is continuous with respect to q.

If the sequence $q_1 \leq \ldots \leq q_n \leq \ldots$ consists of algebra seminorms then the transition maps $\phi_{q_{n+1}}^{q_n}$ are algebra homomorphisms and

$$A \xrightarrow{\cong} \varprojlim_{n \in \mathbf{N}} A_{q_n}$$

is an isomorphism of Fréchet algebras.

Fréchet-Stein algebras cont'd

Definition 8. The K-Fréchet algebra A is called a K-Fréchet-Stein algebra if there is a sequence $q_1 \leq \ldots \leq q_n \leq \ldots$ of continuous algebra seminorms on A which define the Fréchet topology such that

- A_{q_n} is (left) noetherian, and
- A_{q_n} is flat as a right $A_{q_{n+1}}$-module (via $\phi_{q_{n+1}}^{q_n}$)

for any $n \in \mathbf{N}$.

The rings $D(\mathbf{Z}_p, K) = \mathcal{O}(\mathbf{B})$ and $D(o_L, K) = \mathcal{O}(\hat{o}_L)$ are Fréchet-Stein algebras with the seminorms (actually norms) coming from the affinoid algebras making up the rigid structure on the underlying rigid Stein varieties.

Coherent Sheaves

We copy notions from the commutative case of functions on a Stein space.

Definition 9. A coherent sheaf for $(A, (q_n))$ is a family $(M_n)_{n \in \mathbf{N}}$ of modules M_n in $\mathcal{M}_{A_{q_n}}$ together with isomorphisms $A_{q_n} \otimes_{A_{q_{n+1}}} M_{n+1} \xrightarrow{\cong} M_n$ in $\mathcal{M}_{A_{q_n}}$ for any $n \in \mathbf{N}$.

The category $Coh_{(A,(q_n))}$ of coherent sheaves over a Fréchet-Stein algebra is abelian, with the obvious notions of (co)kernels and (co)images. (Flatness is key)

Coherent Sheaves

For any coherent sheaf $(M_n)_n$ for $(A, (q_n))$ its A-module of "global sections" is defined by

$$\Gamma(M_n) := \varprojlim_n M_n.$$

Definition 10. A (left) A-module is called *coadmissible* if it is isomorphic to the module of global sections of some coherent sheaf for $(A, (q_n))$.

Finitely generated modules over Noetherian Banach algebras are well behaved, and this is the foundation for making the Coherent sheaves a good category.

Noetherian Banach Algebras

Theorem 11. *Suppose that A is a (left) noetherian K-Banach algebra; we then have:*

- *Each finitely generated A-module M carries a unique K-Banach space topology (called its canonical topology) such that the A-module structure map $A \times M \to M$ is continuous;*
- *every A-submodule of a finitely generated A module is closed in canonical topology; in particular, every (left) ideal in A is closed;*
- *any module homomorphism between finitely generated A-modules is continuous and strict for the canonical topologies.*

Coadmissible modules

The full subcategory $\mathcal{C}_A$ of coadmissible modules in the category Mod(A) is independent of the choice of the sequence $(q_n)_n$.

Passing to global sections defines a functor

$$\Gamma : Coh_{(A,(q_n))} \longrightarrow \mathcal{C}_A \ .$$

Theorem 12. *Let $(M_n)_n$ be a coherent sheaf for $(A, (q_n))$ and put $M := \Gamma(M_n)$; we have:*

1. *(Theorem A) For any $n \in \mathbf{N}$ the natural map $M \longrightarrow M_n$ has dense image with respect to the canonical topology on the target;*
2. *(Theorem B)*
$$\varprojlim_n{}^{(i)} M_n = 0$$
for any natural number $i \geq 1$. Expressed differently: the projective limit functor from coherent sheaves to modules is exact.

Consequences of Theorems A and B

It follows from Theorems A and B that: If $M := \Gamma(M_n)$ for some coherent sheaf for an F-S algebra A, the natural map

$$A_{q_n} \otimes_A M \xrightarrow{\cong} M_n$$

is an isomorphism for any $n \in \mathbf{N}$. Consequently, the categories $Coh_{(A,(q_n))}$ of coherent sheaves and $\mathcal{C}_A$ of coadmissible modules are equivalent.

Properties of coadmissible modules

1. The direct sum of two coadmissible A-modules is coadmissible;
2. the (co)kernel and (co)image of an arbitrary A-linear map between coadmissible A-modules are coadmissible;
3. the sum of two coadmissible submodules of a coadmissible A-module is coadmissible;
4. any finitely generated submodule of a coadmissible A-module is coadmissible;
5. any finitely presented A-module is coadmissible.

In particular, $\mathcal{C}_A$ is an abelian subcategory of Mod(A).

The canonical topology

Write

$$M = \varprojlim_n M_n \ .$$

Each M_n carries its canonical Banach space topology as a finitely generated A_{q_n}-module.

With the projective limit topology M becomes a K-Fréchet space. Moreover, the A-module structure map $A \times M \to M$ clearly is continuous. This topology is called the *canonical topology* on M.

Algebra and topology

For any coadmissible A-module M and any submodule $N \subseteq M$ the following assertions are equivalent:

1. N is coadmissible;
2. M/N is coadmissible;
3. N is closed in the canonical topology of M.
4. Any A-linear map $f : M \to N$ between coadmissible A-modules is continuous and strict for the canonical topologies;
5. in particular, any finitely generated left ideal of A is closed.

These properties follow from the passing to the sheaf case, using the fact that the Banach algebras defining A are Noetherian, and that the global section functor is exact.

9 Analytic Admissibility

Main Theorem on Distribution Algebras

Theorem 13. *Let G be a compact locally L-analytic group and let K be a discretely valued spherically complete field extension of L. Then $D(G, K)$ is a Fréchet-Stein algebra.*

Admissible analytic representations

This theorem allows us to make the following key definition.

Definition 14. A locally analytic G-representation on a vector space V is called *admissible* if V is of compact type and the strong dual V'_b is a coadmissible $D(H, K)$-module with its canonical topology for one (or all) compact open subgroups of H of G.

The fact that we may check the condition for one compact-open only will become clearer in the course of the proof that we will outline.

Good properties of the category

The properties of coadmissible modules tell us that:

- The category $\mathrm{Rep}_K^a(G)$ of admissible G-representations is an abelian category.
- kernels and image are the algebraic kernel and image (as subspaces).
- Any map in the category is strict and has closed image.
- If V is admissible, so is any closed G-invariant subspace.

The locally analytic principal series are admissible

The locally analytic principal series representations $\mathrm{Ind}_P^G(\chi)$ for $G = \mathrm{GL}_n(L)$ are admissible.

By the Iwasawa decomposition $G = G_o P$ where $G_o = \mathrm{GL}_n(o_L)$, any function $f \in \mathrm{Ind}_P^G(\chi)$ is determined by its restriction to G_o

As a representation space for G_o we have

$$\mathrm{Ind}_P^G(\chi) = \mathrm{Ind}_{P \cap G_o}^{G_o}(\chi|_{P \cap G_o}).$$

The right side is a closed subspace of $C^{an}(G_o, K)$, so its dual is a (Hausdorff) quotient of $D(G_o, K)$. Consequently the dual is coadmissible and therefore the principal series is admissible.

Part II

Admissibility and Analytic Vectors

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10 Proof of the main theorem

Sketch of proof of main theorem on distribution algebras

Theorem 24.1: *Let G be a compact locally L-analytic group, and suppose that K is discretely valued. Then $D(G, K)$ is a Fréchet-Stein algebra.*

The proof of this result relies on two main ingredients:

1. The theory of p-valued groups (esp. uniform groups) as introduced in Dat's lectures. This allows us to put explicit coordinates on $D(G, K)$ and define the putative F-S structure.

2. The theory of filtered and graded rings, also discussed by Dat. This allows us to prove the necessary properties of the F-S structure.

Uniform groups

Recall from Dat's lecture that uniform groups are a particularly simple class of pro-p groups, and that any compact $\mathbf{Q}_p$-analytic group has a compact open normal uniform subgroup.

Uniform groups carry a p-valuation and have a good set of multiplicative coordinates.

Coordinates on uniform groups

Theorem 15. *As discussed in Dat's lecture, any uniform group H has an (ordered) set $h_1, \ldots, h_d$ of topological generators such that:*

1. *The map*

$$\begin{array}{rcc} \psi : \mathbf{Z}_p^d & \to & H \\ (x_1, \ldots, x_d) & \to & h_1^{x_1} \cdots h_d^{x_d} \end{array}$$

is a global chart for the manifold H.

2. The p-valuation ω is defined by

$$\omega(h_1^{x_1}\cdots h_d^{x_d}) = \inf_{1\le i\le d}(1+\omega(x_i)).$$

Explicit representation of distributions

The chart ψ gives us explicit coordinates for $C^{an}(H,K)$ and $C(H,K)$:

$$\psi^* : C^{an}(H,K) \xrightarrow{\cong} C^{an}(\mathbf{Z}_p^d, K)$$

Mahler tells us that $C(\mathbf{Z}_p^d, K)$ can be viewed as the space of all series

$$f(x) = \sum_{\alpha\in\mathbf{N}_0^d} c_\alpha \binom{x}{\alpha}$$

with $c_\alpha \in K$ and such that $|c_\alpha| \longrightarrow 0$ as $|\alpha| \longrightarrow \infty$.

Here $\binom{x}{\alpha}$ is shorthand for a product of binomial functions.

Explicit representation of distributions cont'd

Amice's theorem says that the Mahler expansion of f lies in the subspace $C^{an}(\mathbf{Z}_p^d, K)$ if and only if $|c_\alpha| r^{|\alpha|} \longrightarrow 0$ for some real number $r > 1$ as $|\alpha| \longrightarrow \infty$.

For $h = \psi(x)$ we have

$$h(f) = \delta_{\psi(x)}(f) = \psi^*(f)(x)$$

for any $f \in C(H,K)$ and any $x \in \mathbf{Z}_p^d$.

Of course, f is continuous if the coefficients c_α go to zero as $|\alpha| \to \infty$.

Explicit representation of distributions cont'd

Write $b_i := h_i - 1$ and $\mathbf{b}^\alpha := b_1^{\alpha_1} b_2^{\alpha_2} \cdots b_d^{\alpha_d}$, for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{N}_0^d$.

We have

$$\mathbf{b}^\alpha \in \mathbf{Z}_p[H] \subseteq D(H,K).$$

If c_α denote the coefficients of the Mahler expansion of $\psi^*(f)$ for some $f \in C(H,K)$ then

$$\mathbf{b}^\alpha(f) = c_\alpha\ .$$

Explicit representation of distributions cont'd

Any locally analytic distribution $\lambda \in D(H,K)$ has a unique convergent expansion

$$\lambda = \sum_{\alpha\in\mathbf{N}_0^d} d_\alpha \mathbf{b}^\alpha$$

with $d_\alpha \in K$ such that, for any $0 < r < 1$, the set $\{|d_\alpha| r^{|\alpha|}\}_{\alpha \in \mathbf{N}_0^d}$ is bounded. Conversely, any such series is convergent in $D(H, K)$. The Fréchet topology on $D(H, K)$ is defined by the family of norms

$$\|\lambda\|'_r := \sup_{\alpha \in \mathbf{N}_0^d} |d_\alpha| r^{\kappa|\alpha|}$$

for $0 < r < 1$. These generalize Lazard's study of the case $r = 1/p$.

The continuous distributions have expansions with bounded coefficients.

As a vector space, $D(H, K)$ (and $D^c(H, K)$) looks like $D(\mathbf{Z}_p^d, K)$ (or $D^c(\mathbf{Z}_p^d, K)$), but the multiplication is twisted up.

The Fréchet-Stein structure

As Dat has discussed, Lazard proves in his work on pro-p groups that:

1. The norm $\| \ \|_{1/p}$ induces the compact topology on the ring $\mathbf{Z}_p[[H]] \subset D(H, K)$.
2. The norm $\| \ \|_{1/p}$ is multiplicative and independent of the choice of ordered basis h_i.
3. The norm $\| \ \|_{1/p}$ satisfies $\|g - 1\|_{1/p} \leq p^{-\omega(g)}$

The Fréchet-Stein structure cont'd

From Lazard's results we can establish that each norm $\| \ \|_r$, for $1/p \leq r < 1$, is submultiplicative on $D(H, K)$.

The following lemma is crucial later:

Lemma 16. *When* $1/p < r < 1$,

$$\|b_i b_j - b_j b_i\|_r < \|b_i b_j\|_r.$$

We will indicate the proof of this lemma.

The Fréchet-Stein structure cont'd

Since $i < j$ we have $\|b_i b_j\|_r = r^{2\kappa}$ and κ is 1 or 2 as p is odd or even respectively.

The properties of a p-valuation imply that $h := h_i^{-1} h_j^{-1} h_i h_j = g^p$ for some $g \in H$.

Therefore

$$\begin{aligned} b_i b_j - b_j b_i &= h_i h_j - h_j h_i \\ &= = h_j h_i (h - 1) \\ &= h_j h_i (g^p - 1) \\ &= h_j h_i (((g - 1) + 1)^p - 1) \\ &= h_j h_i (g - 1)^p + \sum_{n=1}^{p-1} \binom{p}{n} h_j h_i (g - 1)^n \end{aligned}$$

so by submultiplicativity we have

$$\|b_i b_j - b_j b_i\|_r \leq \max(\|g-1\|_r^p, |p|\|g-1\|_r) \ .$$

The Fréchet-Stein structure cont'd

Lazard's results imply that $\|g-1\|_r^p \leq r^{p\omega(g)}$ so we have $\|b_i b_j - b_j b_i\|_r \leq \max(r^{p\omega(g)}, p^{-1} r^{\omega}(g))$.

The properties of a p-valuation imply that $\omega(g) \geq \omega(h_i) + \omega(h_j) - 1$. The desired results follow from the inequalities

$$\begin{array}{rcl} r^{p(\omega(h_1)+\omega(h_2)-1)} & < & r^{2\kappa} \\ r^{\omega(h_1)+\omega(h_2)-1}/p & < & r^{2\kappa} \end{array}$$

Define

1. $D_r(H,K) = \{f = \sum d_\alpha \mathbf{b}^\alpha : \|d_\alpha\| r^{|\alpha|} \to 0 \text{ as } |\alpha| \to \infty\}$. This is the completion of $D(H,K)$ with respect to $\| \ \|_r$.
2. $D_{<r}(H,K) = \{f = \sum d_\alpha \mathbf{b}^\alpha : |d_\alpha| r^{|\alpha|} \text{ is bounded as } |\alpha| \to \infty\}$

The Fréchet-Stein structure continued

We have

$$D_r(H,K) \subseteq D_{<r}(H,K) \subseteq D_{r'}(H,K) \subseteq \ldots \subseteq D_{1/p}(H,K)$$

when $1/p \leq r' < r < 1$. Also

$$D(H,K) = \operatorname{proj\,lim} D_r(H,K) = \operatorname{proj\,lim} D_{<r}(H,K).$$

We will see that the sequence $D_r(H,K)$ give a Fréchet-Stein structure on $D(H,K)$.

Filtrations and norms

We apply the properties of zariskian filtrations described by Dat to show that $D(H,K)$ is a Fréchet-Stein algebra.

The algebras $D_r(H,K)$ for each $1/p \leq r' \leq r$, have the filtrations:

$$F^s_{r'} D_r(H,K) = \{a \in D_r(H,K) : \|a\|_{r'} \leq p^{-s}\}.$$

The same filtrations make sense for $D_{<r}(H,K)$ provided that $1/p \leq r' < r$. Also, all of these filtrations coincide on K with the natural one.

Computations with graded rings

A computation using the explicit coordinates for $1/p \leq r < 1$ shows there are isomorphisms

$$gr^{\cdot} K \otimes_{gr^{\cdot} \mathbf{Z}_p} gr^{\cdot}_r \mathbf{Z}_p[[G]] \to gr^{\cdot}_r K[[G]] \to gr^{\cdot}_r D_r(G, K).$$

As Dat discussed, Lazard computed $gr^{\cdot}_{1/p} \mathbf{Z}_p[[G]]$ and showed that it is a noetherian integral domain, but not in general commutative.

Computations of graded rings

Theorem 17. *For $1/p < r < 1$, and $r \in p^{\mathbf{Q}}$ the ring $gr^{\cdot}_r D_r(H, K)$ is a (commutative) polynomial ring over $gr^{\cdot} K$ in the principal symbols $\sigma(b_i)$ for $i = 1, \ldots, d$.*

Proof.

- $gr^{\cdot}_r D_r(H, K)$ is a free $gr^{\cdot} K$ module generated by the principal symbols of the monomials $\sigma(\mathbf{b}^\alpha)$.
- $$\sigma(\mathbf{b}^\alpha) = \prod_{i=1}^{d} b_i^{\alpha_i}$$
- Recall that we have shown earlier that, when $i < j$:
  $$\|b_i b_j - b_j b_i\|_r < \|b_i b_j\|_r.$$

These facts imply that the $\sigma(b_i)$ are commuting variables and this yields the claimed result. □

Proof of Fréchet-Stein Property

We see that, since the filtration we have defined on $D_r(H, K)$ is separated and complete, with noetherian associated graded ring, that the Banach algebras $D_r(H, K)$, for $1/p < r < 1$, and $r \in p^{\mathbf{Q}}$, are Noetherian.

Theorem 18. *For $1/p < r' \leq r < 1$ in $p^{\mathbf{Q}}$, the map $D_r(H, K) \to D_{r'}(H, K)$ is flat.*

First consider $D_r(H, K) \to D_{<r}(H, K)$.

1. Check the result after extending K so that $r = |\pi|^m$ and $r' = |\pi|^{m'}$ are integral powers of $|\pi|$, where π is the uniformizer of K.
2. Calculate
   $$gr^0_r D_{<r}(H, K) = k[[u_1, \ldots, u_d]]$$
   where u_i is the principal symbol of b_i/π^m. Then
   $$gr^{\cdot}_r D_{<r}(H, K) = gr^{\cdot} K \otimes gr^0_r D_{<r}(H, K)$$
   is a noetherian ring.
3. Check that $gr^0_r D_r(H, K)$ is the subring of polynomials $k[u_1, \ldots, u_d]$.

Thus on the level of graded rings the map from $D_r(H, K)$ to $D_{<r}(H, K)$ is a flat base change of the map of a polynomial ring into formal power series, and this is known to be flat.

Proof of Fréchet-Stein Property cont'd

Next consider $D_{<r}(H,K) \to D_{r'}(H,K)$.

Show that

$$F^0_{r'} D_{<r}(H,K) \to D_{r'}(H,K)$$

is flat. The key point here is that the image of $F^0_{r'} D_r(H,K)$ in the target is compact (at least if K is locally compact; more sophisticated methods work if K is only discretely valued).

Therefore the filtration $F_{r'}$ on $F^0 D_r(H,K)$ is complete.

A computation shows that the map on graded rings is given by "localization at $\sigma(\pi)$" and is therefore flat.

Completion of the proof of the main theorem

We've shown that $D(H,K)$ is Fréchet-Stein for a uniform $\mathbf{Q}_p$-analytic group H. To pass to the general case requires two steps.

First, if G is compact, it has a normal compact open subgroup H that is uniform. Then $D(G,K) = \oplus D(H,K)\delta_{g_i}$ for a family of coset representatives of G/H. From the norms $\| \, \|_r$ on $D(H,K)$ we construct a family of norms on $D(G,K)$ by taking the supremum over the coordinates. This gives a Fréchet-Stein structure on $D(G,K)$.

If G is L-analytic, then $D(G,K)$ is a quotient of $D(G_0,K)$ by the ideal generated by the Lie algebra $\mathfrak{g}$. A general property of Fréchet-Stein algebras says that $D(G,K)$ is then Fréchet-Stein and the coadmissible modules for $D(G,K)$ are precisely the $D(G,K)$-modules that are coadmissible as $D(G_0,K)$-modules.

11 Analytic vectors

Analytic vectors in continuous representations

Suppose that V is a locally convex Hausdorff K-vector space with a continuous G action, where G is a locally L-analytic group.

Definition 19. A vector $v \in V$ is called (locally)-analytic if the orbit map $g \mapsto gv$ belongs to $C^{an}(G,V)$.

We are interested in the existence of analytic vectors in continuous G-representations V, particularly in the case when V is an admissible Banach space representation.

Definition 20. Let V be a K-Banach space representation of G (in the sense of Dat's lectures). Then we define V_{an} to be the subspace of V consisting of analytic vectors, equipped with the topology induced by the embedding $V_{an} \hookrightarrow C^{an}(G,V)$ given by $v \mapsto (g \mapsto g^{-1}v)$.

Emerton's analytic vectors

Emerton has given a more general, and more intrinsic, definition of analytic vectors constructed as the convex direct limit of subspaces of V indexed by subgroups H of G consisting of vectors whose orbit maps are "rigid-analytic" with respect to H.

The two definitions give the same set of analytic vectors, but Emerton's space has a finer topology.

When V is an admissible Banach space representation, Emerton's construction of V_{an} is equivalent to the one introduced here.

12 Analytic vectors: the $\mathbf{Q}_p$-analytic case

Existence of analytic vectors

The first part of this lecture is devoted to proving the following theorem:

Theorem 21. *Let G be a compact locally $\mathbf{Q}_p$-analytic group and let V be an admissible Banach space representation of G. Then*

1. *V_{an} is dense in V.*
2. *V_{an} is an admissible locally analytic G-representation.*
3. *$V'_{an} = D(G,K) \otimes_{K[[G]]} V'$ where $K[[G]] = D^c(G,K)$ is the continuous distribution algebra for G.*

Moreover, the functor $V \mapsto V_{an}$ is exact.

Proof of existence of analytic vectors

The first step in the proof is to establish that V_{an} is an admissible G-representation. This does not rely on the $\mathbf{Q}_p$-analyticity hypothesis.

V being finitely generated over $K[[G]]$ means that there is a surjection

$$K[[G]]^m \to V'$$

and thus a continuous embedding

$$V \hookrightarrow C(G,K)^m \text{ for some integer } m.$$

From this one establishes the existence of a closed embedding

$$V_{an} \hookrightarrow C^{an}(G,K)^m$$

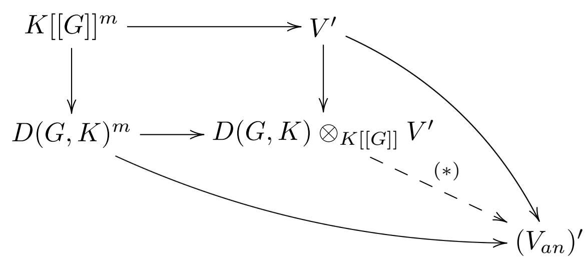
showing V_{an} to be of compact type. Dually one has a continous surjection

$$D(G,K)^m \to (V'_{an})_b$$

proving that $(V_{an})'_b$ is coadmissible and carries its canonical topology.

Existence of analytic vectors cont'd

The second step is to show that $V_{an} = D(G,K) \otimes_{K[[G]]} V'$. Consider:



- If we equip $K[[G]]$ and V' with the "bounded weak" topology (used in the theory of continuous representation) then the vertical maps in the square are continuous and have dense image.
- The map $D(G,K) \otimes_{K[[G]]} V' \to V'_{an}$ is continuous if both spaces have canonical (i.e. strong) topologies.

Existence of analytic vectors cont'd

Looking at strong duals and using density of the images of maps in the previous slide, we consider

$$V_{an} \to (D(G,K) \otimes_{K[[G]]} V')'_b \to ((V'_{bs})')_b$$

- The maps are continuous by functoriality.
- Both maps are injective because dual maps are surjective or have dense image.
- The middle space is a locally analytic G-representation.
- $(V'_{bs})'_b$ is the Banach space V. (This is Schikof's duality discussed by Dat, expressed slighly differently).

Since V_{an} consists of all analytic vectors in V, this forces the maps to be surjections.

Existence of analytic vectors cont'd

To complete the proof, consider the passage from a continuous admissible representation V to V_{an} by the route

$$V$$
$$\downarrow$$
$$\text{finitely generated } K[[G]]\text{-module } V'$$
$$\downarrow$$
$$\text{coadmissible module } D(G,K) \otimes_{K[[G]]} V'$$
$$\downarrow$$
$$(D(G,K) \otimes_{K[[G]]} V')'_b = V_{an}$$

The key step is the middle one.

Faithful flatness

Our theorem follows from the following algebraic theorem.

Theorem 22. *Let G be a compact locally L-analytic group and let G_0 be the associated $\mathbf{Q}_p$-analytic group. Then the map $L[[G]] \to D(G_0, K)$ is faithfully flat.*

Indeed, the exactness of the functor $V \to V_{an}$ follows from flatness. The density of V_{an} in V follows from injectivity of the map

$$V' \to D(G, K) \otimes_{K[[G]]} V'$$

which is a consequence of faithful flatness.

Faithful flatness cont'd

Suppose H_0 in G_0 is a uniform compact open normal subgroup. Then $D(G_0, K) = L[[G]] \otimes_{L[[H_0]]} D(H_0, K)$ and the faithful flatness of $D(G_0, K)$ follows from that of $D(H_0, K)$ by base change. Thus we may assume that G is uniform.

Next we show that the maps

$$\mathbf{Z}_p[[G]] \to L[[G]] \to D_r(G, K)$$

are all flat.

Faithful flatness cont'd

We fall back on the filtered ring methods discussed by Dat and used previously. Recall the isomorphisms

$$gr^{\cdot} K \otimes_{gr^{\cdot} \mathbf{Z}_p} gr^{\cdot}_r \mathbf{Z}_p[[G]] \to gr^{\cdot}_r K[[G]] \to gr^{\cdot}_r D_r(G, K).$$

On the graded level the map $\mathbf{Z}_p[[G]] \to D_r(G, K)$ is base change by the map $gr^{\cdot} \mathbf{Z}_p \to gr^{\cdot} K$ which is flat.

The ring $L[[G]] = L \otimes \mathbf{Q}_p[[G]]$ has $\mathbf{Q}_p[[G]]$ as a direct summand.

$D_r(G, K) \otimes_{L[[G]]} M$ is a direct summand of $D_r(G, K) \otimes_{\mathbf{Z}_p[[G]]} M$ when M is an $L[[G]]$ module.

Therefore the map $L[[G]] \to D_r(G, K)$ is flat.

Faithful flatness cont'd

To show that the map $L[[G]] \to D(G, K)$ is flat, let J be a left ideal of $L[[G]]$.

- $L[[G]]$ is (left) Noetherian so J is finitely presented and $D(G, K) \otimes_{L[[G]]} J$ is coadmissible.

- $$D(G,K)\otimes_{L[[G]]} J = \operatorname{proj\,lim}_r (D_r(G,K)\otimes_{D(G,K)} (D(G,K)\otimes_{L[[G]]} J)).$$

- The flatness of $D_r(G,K)$ over $L[[G]]$ implies that $D_r(G,K)\otimes_{L[[G]]} J$ injects into $D_r(G,K)$.

- Left-exactness of the projective limit implies that $D(G,K)\otimes_{L[[G]]} J$ injects into $D(G,K)$.

This implies the desired flatness (and exactness of the functor $V \to V_{an}$.

Faithful flatness cont'd

Proof of faithful flatness requires establishing that $D(G,K)\otimes_{L[[G]]} M \neq 0$ for non-zero $L[[G]]$ modules M.

This in turn reduces to showing that, for any proper left ideal J in $o[[G]]$ such that $o[[G]]/J$ is p-torsion free, there is an r such that $D_r(G,K)J$ is a proper submodule of $D_r(G,K)$.

We omit this rather technical argument using filtrations for now.

Results of Kohlhaase and Schmidt shed light on the existence of analytic vectors in the L-analytic case. Before returning to this topic we develop some additional theory in the L-analytic case.

13 The hyperenveloping algebra

Support of distributions

Definition 23. Let M be a locally L-analytic manifold. The support of a distribution $\delta \in D(M,K)$ is the complement of the largest open set U of M such that δ vanishes on functions f supported on U. If $C \subset M$ and $V \subset D(M,K)$ (resp. $C^{an}(M,K)$) then we write V_C for the subspace of distributions supported in C (resp. the subspace of functions supported in C).

Suppose C is compact in $D(M,K)$. Define, for open sets U containing C,

$$C^{\omega}_C(M,K) = \varinjlim_{U \supset C} C^{an}(U,K)$$

We have

$$D(M,K)_C \xrightarrow{\sim} C^{\omega}_C(M,K)'_b$$

where both spaces are nuclear Fréchet spaces.

Canonical coordinates of the second kind and germs of functions

Suppose that G is a locally L-analytic group and let $\mathfrak{x}_1, \ldots, \mathfrak{x}_d$ be an ordered L-basis for $\mathfrak{g}$. The map

$$(t_1, \ldots, t_d) \to \exp(t_1\mathfrak{x}_1) \cdot \exp(t_d\mathfrak{x}_d)$$

for $(t_1, \ldots, t_d)$ sufficiently close to 0 in $\mathfrak{g}$, gives a chart for a neighborhood of 1 in G. (These are called "canonical coordinates of the second kind".)Write $\alpha = (\alpha_1, \ldots, \alpha_d)$ with the $\alpha_i \geq 0$ in $\mathbf{Z}$ and $|\alpha| = \sum_{i=1}^{d} \alpha_i$.

Let

$$\mathbf{T}^\alpha(\exp(t_1\mathfrak{x}_1) \cdot \exp(t_d\mathfrak{x}_d)) = t_1^{\alpha_1} \cdots t_d^{\alpha_d}.$$

In these coordinates, the space $C_1^\omega(G, K)$ is

$$\left\{ \sum_\alpha c_\alpha \mathbf{T}^\alpha | c_\alpha \in K \text{ and for some } r > 0 |c_\alpha| r^{|\alpha|} \to 0 \text{ as} |\alpha| \to \infty \right\}$$

The hyperenveloping algebra

In general, if G is a locally L-analytic group, and $C \subset G$ is compact, contains $1 \in G$ and satisfies $C \cdot C \subset C$, then $D(G, K)_C$ is a closed nuclear Fréchet subalgebra of $D(G, K)$.

The most important case of all is the "hyperenveloping algebra" $D(G, K)_1$ of distributions supported at the identity.

$D(G, K)_1$ is dual to the space $C^\omega_{\{1\}}(G, K)$ of germs of L-analytic functions at the identity.

We can describe $D(G, K)_1$ explicitly.

The hyperenveloping algebra cont'd

The action of the universal enveloping algebra $U(\mathfrak{g})$ on $C^{an}(G, K)$ as distributions supported at the identity gives an embedding $U(\mathfrak{g}) \otimes_L K \hookrightarrow D(G, K)$ whose image lands in $D(G, K)_1$.

The distribution

$$\mathfrak{X}^\alpha = \mathfrak{x}_1^{\alpha_1} \cdots \mathfrak{x}_d^{\alpha_d}$$

is dual to $\mathbf{T}^\alpha$ by Taylor's theorem. Thus

$$D(G, K)_1 = C_1^\omega(G, K)'_b =$$

$$\left\{ \sum_\alpha d_\alpha \mathfrak{X}^\alpha | d_\alpha \in K \text{ and, for all } r > 0, \sup |d_\alpha| r^{-|\alpha|} < \infty \right\}$$

The hyperenveloping algebra cont'd

$D(G,K)_1$ "looks like" a space of entire power series (though it is non-commutative.)

$U(\mathfrak{g})\otimes_L K$ (the subspace of polynomials) is dense in this space. So $D(G,K)_1$ is the closure of $U(\mathfrak{g})\otimes K$. We write

$$D(G,K)_1 = U(\mathfrak{g},K)$$

to emphasize that we are dealing with a completed version of $U(\mathfrak{g})$.

The algebra $U(\mathfrak{g},K)$ proves to be a Fréchet-Stein algebra that is simpler that $D(G,K)$ but closely related to it. To explore this we need to refine the notion of uniform group to account for L-analyticity.

Uniform* groups

Suppose G is a d-dimensional locally L-analytic group.

G is called uniform* if G_0 is uniform as a $\mathbf{Q}_p$-analytic group and there is minimal ordered set of generators for G_0 of the form

$$h_{ij} = \exp(v_i \mathfrak{x}_j)$$

where $\mathfrak{x}_1,\ldots,\mathfrak{x}_d$ is an L-basis of $\mathfrak{g}$ and $v_1 = 1,\ldots,v_n$ is a basis of o_L over $\mathbf{Z}_p$.

$U(\mathfrak{g},K)$ **inside** $D(G,K)$

The distribution algebra $D(G_0,K)$ is a ring of (non-commutative) power series in the variables $b_{ij} = h_{ij} - 1$.

Tracing back through the definitions involving Mahler expansions and using the fact that, for $\mathbf{Z}_p$, the distribution $f \mapsto \frac{df}{dz}|_{z=0}$ we see that $v_i\mathfrak{x}_j$ as an element of the Lie algebra $\mathfrak{g}_0$ corresponds to the power series $\log(1+b_{ij})$.

Frommer-Kohlhaase theorem

Recall that $D(G,K)$ is the quotient of $D(G_0,K)$ by the ideal I generated by the kernel $\mathfrak{g}^0$ of the map

$$L\otimes_{\mathbf{Q}_p}\mathfrak{g}\to\mathfrak{g}.$$

In the current terms, this is the ideal generated by the elements

$$F_{ij} = \log(1+b_{ij}) - v_i\log(1+b_{1j})$$

in $D(G_0,K)$. (Remember the second fundamental example from Lecture I).

Let $\|\cdot\|_r$ be the norm on $D(G_0,K)$ and the quotient norm it induces on $D(G,K)$. By carefully studying the expansion of the function $\log(1+x)$, Kohlhaase generalized a result of Frommer in the $\mathbf{Q}_p$-case to obtain the following theorem.

Frommer-Kohlhaase Theorem cont'd

Theorem 24. *Let G be a uniform* group with generators h_{ij} and with $b_{ij} = h_{ij} - 1$. Let $U_r(\mathfrak{g}, K)$ be the closure of $U(\mathfrak{g}, K)$ in $D_r(G, K)$. Then there is a positive integer $\epsilon(r, p)$ such that $D_r(G, K)$ is finite free over $U_r(\mathfrak{g}, K)$ on the basis consisting of those monomials $\mathbf{b}^\alpha$ where $\alpha_i < \epsilon(r, p)$. As a vector space*

$$U_r(\mathfrak{g}, K) = \left\{ \sum_\alpha d_\alpha \mathfrak{X}^\alpha \text{ where} \|d_\alpha \mathfrak{X}^\alpha\|_r \to 0 \text{ as} |\alpha| \to 0 \right\}$$

Finally $U(\mathfrak{g}, K)$ is Fréchet-Stein for the family of algebras $U_r(\mathfrak{g}, K)$.

The Frommer-Kohlhaase Theorem cont'd

It is worth remarking that the key step in establishing flatness of the maps $U_r(\mathfrak{g}, K) \to U_{r'}(\mathfrak{g}, K)$ is to consider the diagram

$$\begin{array}{ccc} U_r(\mathfrak{g}, K) & \longrightarrow & U_{r'}(\mathfrak{g}, K) \\ \downarrow & & \downarrow \\ D_r(G, K) & \longrightarrow & D_{r'}(G, K) \end{array}$$

The lower map is flat and the vertical maps are finite free ring extensions.

The hyperenveloping algebra cont'd

One can say more about the Fréchet-Stein structure of $U(\mathfrak{g}, K)$ and the associated filtrations and graded rings.

Theorem 25. *Suppose G is uniform* with $\mathfrak{x}_1, \ldots, \mathfrak{x}_d$ the relevant L-basis of $\mathfrak{g}$. Then the norm $\| \, \|_r$ on $D(G, K)$ induces a filtration on $U(\mathfrak{g}, K)$ such that*

$$gr^{\cdot}_r U_r(\mathfrak{g}, K) \to gr^{\cdot}_r D_r(G, K)$$

is finite and free on the principal symbols of the basis in the Frommer-Kohlhaases theorem. Furthermore, $gr^{\cdot}_r U_r(\mathfrak{g}, K)$ is a polynomial ring in the principal symbols $\sigma(\mathfrak{x}_i)$ and the norm $\| \, \|_r$ is multiplicative on $U_r(\mathfrak{g}, K)$. Finally, the monomials $\mathfrak{X}^\alpha$ are an orthogonal basis for $U_r(\mathfrak{g}, K)$ so that

$$\| \sum_\alpha d_\alpha \mathfrak{X}^\alpha \|_r = \sup_\alpha |d_\alpha| \|\mathfrak{X}\|_r^\alpha$$

The hyperenveloping algebra cont'd

We emphasize that in the L-analytic case the graded rings associated to $D_r(G, K)$ are not in general polynomial rings but involve equations of the form $x^{p^n} - y^{p^n}$.

Thus $U(\mathfrak{g}, K)$ is much simpler to work with.

The next slide gives a general result on compatibility of uniform* structures that is proved (in part) using this approach.

Compatible uniform* structures

Theorem 26. *Let G be a compact locally L-analytic group and let P be a closed subgroup of G of dimension $l \leq d$.*

- *There is an open normal subgroup G' in G that is uniform* with respect to bases $\mathfrak{x}_1, \ldots, \mathfrak{x}_d$ and $v_1, \ldots, v_n$ of $\mathfrak{g}/L$ and $o_L/\mathbf{Q}_p$ respectively.*
- *These may be chosen so that $\mathfrak{x}_1, \ldots, \mathfrak{x}_l$ and $v_1, \ldots, v_n$ make $P' = P \cap G'$ uniform*.*
- *One may choose norms $\| \|_r$ on $D(G_0, K)$ that give Fréchet-Stein structures compatibly on all of $D(G_0, K)$, $D(G, K)$, $D(P_0, K)$, and $D(P, K)$.*
- *With these structures $D_r(G_0, K)$ is flat over $D_r(P_0, K)$ and $D_r(G, K)$ is flat over $D_r(P, K)$.*

The flatness assertion is proved by reduction to the hyperenveloping algebras and exploiting the freeness of D_r over U_r.

14 Analytic vectors more generally

Passage to L-analytic vectors

Suppose that G is a compact locally L-analytic group and that V is an admissible Banach space G-representation on a K-vector space. We may find the L-analytic vectors in V in two stages:

1. First take the $\mathbf{Q}_p$ analytic vectors $V_{\mathbf{Q}_p-an}$ for the associated $\mathbf{Q}_p$-analytic group G_0. (This functor is dual to the faithfully flat base change $L[[G]] \to D(G, K)$ and is exact, with $V_{\mathbf{Q}_p-an}$ dense in V).
2. Then take the subspace of L-analytic vectors in $V_{\mathbf{Q}_p-an}$. This process amounts to taking the vectors annihilated by the kernel $\mathfrak{g}^0$ of $L \otimes_{\mathbf{Q}_p} \mathfrak{g} \to \mathfrak{g}$.

Following work of T.Schmidt we focus on the second of these two functors from admissible $\mathbf{Q}_p$-analytic representations of G_0 to admissible L-analytic representations of G:

$$F^L_{\mathbf{Q}_p}(V) = V^{\mathfrak{g}^0}$$

Passage to L-analytic vectors continued

Dually, the functor $F^L_{\mathbf{Q}_p}$ corresponds to base change

$$D(G_0, K) \to D(G, K).$$

Schmidt interprets $F^L_{\mathbf{Q}_p}$ in terms of Lie algebra cohomology. He shows that:

- $R^i F^L_{\mathbf{Q}_p}(V) = Ext^i_{D(G_0,K)}(D(G, K), V)$ as an isomorphism of admissible G representations.

- If $K/\mathbf{Q}_p$ is finite, $R^i F^L_{\mathbf{Q}_p} = 0$ for $i > ([L : \mathbf{Q}_p] - 1)\, dim_L\, G$.
- $R^i F^L_{\mathbf{Q}_p}$ "commutes with locally analytic induction" (where for simplicity we don't give all the definitions and hypotheses.)
- By explicit computation, for a locally analytic principal series representation of GL_n, we have $R^1 F^L_{\mathbf{Q}_p} \mathrm{Ind}^G_P(\chi) \neq 0$ when χ is smooth.

A faithful flatness result in the L-analytic case

We will discuss another theorem of T. Schmidt on analytic vectors. Assume $L/\mathbf{Q}_p$ is Galois. Let G_σ be the base change of the compact L analytic group by $\sigma : L \to L$ in $\mathrm{Gal}(L/\mathbf{Q}_p)$.

Define

$$G_{L/\mathbf{Q}_p} = \Pi_\sigma G_\sigma$$

This is an L-analytic group with Lie algebra $\mathfrak{g}_{L/\mathbf{Q}_p} = L \otimes_{\mathbf{Q}_p} \mathfrak{g}$.

The $\mathbf{Q}_p$-analytic group G_0 maps into $G_{L/\mathbf{Q}_p}$ diagonally.

Faithful flatness in the L-analytic case cont'd

Although the map $L[[G]] \to D(G, K)$ is not flat in general, one has the following result of T. Schmidt.

Theorem 27. *Suppose $K/\mathbf{Q}_p$ is finite and G is compact L-analytic, with $L/\mathbf{Q}_p$ Galois. Then the map*

$$D^c(G, K) \to D(G_{L/\mathbf{Q}_p}, K)$$

is faithfully flat.

Remarks on the proof of Schmidt's theorem

We make a few remarks on the proof of this theorem.

- First one reduces the problem to establishing faithful flatness of the maps $D_r(G_0, K) \to D_r(G_{L/\mathbf{Q}_p}, K)$ for r sufficiently close to 1.
- Then one reduces to the case that G is uniform* and considers a diagram for the hyperenveloping algebras:

$$\begin{array}{ccc} U_r(\mathfrak{g}_0, K) & \longrightarrow & U_r(\mathfrak{g}_{L/\mathbf{Q}_p}, K) \\ \downarrow & & \downarrow \\ D_r(G_0, K) & \longrightarrow & D_r(G_{L/\mathbf{Q}_p}, K) \end{array}$$

- We observe that we are in a situation where we may deduce flatness and faithful flatness by considering the properties on graded rings.

Schmidt's theorem cont'd

Next we apply the fact that there is a topological isomorphism:

$$U(\mathfrak{g}_0, K) \to U(\mathfrak{g}_{L/\mathbf{Q}_p}, K).$$

It is induced by the dual bijection

$$C_1^\omega(G_{L/\mathbf{Q}_p}, K) \to C_1^\omega((G_{L/\mathbf{Q}_p})_0, K) \to C_1^\omega(G_0, K)$$

The first map is the closed embedding of L-analytic functions into $\mathbf{Q}_p$ analytic ones, the second is restriction to the diagonal – all near the identity

On power series, this amounts to taking a power series converging near 1 in $[L : \mathbf{Q}_p]\, dim_L\, G$ L-valued variables and treating the L-valued variables as $\mathbf{Q}_p$-valued ones.

Schmidt's theorem cont'd

We see that $U(\mathfrak{g}_{L/\mathbf{Q}_p}, K)$ has two families of norms on it; one coming from $U(\mathfrak{g}_0, K)$ and one by restriction from $D(G_{L/\mathbf{Q}_p}, K)$.

Comparing, for r close to 1 the upper arrow in this diagram is injective, norm-decreasing and with dense image:

$$\begin{array}{ccc} U_r(\mathfrak{g}_0, K) & \longrightarrow & U_r(\mathfrak{g}_{L/\mathbf{Q}_p}, K) \\ \downarrow & & \downarrow \\ D_r(G_0, K) & \longrightarrow & D_r(G_{L/\mathbf{Q}_p}, K) \end{array}$$

Thus it is an isomorphism on associated graded objects.

Schmidt's theorem cont'd

Finally one looks closely at the diagram of graded objects and applies the Frommer-Kohlhaase theorem (which says that the vertical arrows are finite free extensions on the graded level) to see that the graded lower map is a finite free extension, hence faithfully flat.

Finally one combines faithful flatness of $D^c(G_0, K) \to D(G_0, K)$, the sheaf-theoretic properties of coadmissible modules, and the faithful flatness just proved of $D_r(G_0, K) \to D_r(G_{L/\mathbf{Q}_p}, K)$ to complete the proof.

15 Irreducibility of Principal Series

The locally analytic principal series

Let $\mathbf{G}$ be a connected reductive group over L and let $\mathbf{P}$ be a parabolic subgroup.

Let (V, ρ) be a finite dimensional locally L-analytic representation of $\mathbf{P}(L)$ on which the unipotent radical of $\mathbf{P}$ acts trivially.

Set

$$\mathrm{Ind}_{\mathbf{P}(L)}^{\mathbf{G}(L)}(\rho) = \{f \in C^{an}(\mathbf{G}(L), V) : f(gp) = \rho(p)^{-1} f(g) \text{ for } p \in \mathbf{P}(L)\}$$

Such representations are called (L-locally-analytic) principal series.

Orlik-Strauch Theorem

Generalizing work for $\mathrm{GL}_2(\mathbf{Q}_p)$ by S-T and by Frommer for $\mathrm{GL}_n(\mathbf{Q}_p)$, Orlik-Strauch have proved the following result.

Theorem 28. *The locally analytic principal series representation* $\mathrm{Ind}_{\mathbf{P}(L)}^{\mathbf{G}(L)}(\rho)$ *is topologically irreducible if the generalized Verma module* $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V'$ *is irreducible as a* $U(\mathfrak{g})$ *module.*

Problem: There is no general recipe for describing the structure of these representations when the Verma module is reducible. Also, Schneider has conjectured more general irreducibility as Dat has remarked.

Example: Principal series for $\mathrm{GL}_2(\mathbf{Q}_p)$

Let $G = \mathrm{GL}_2(\mathbf{Q}_p)$ and P be the (lower triangular) Borel subgroup. Let:

- $\chi : P \to K^*$ be a locally analytic character trivial on unipotents.
- $c(\chi) \in K$ be such that

$$\chi(\begin{matrix} t^{-1} & 0 \\ 0 & t \end{matrix}) = \exp(c(\chi)\log(t)).$$

The Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} K_{-\chi}$ is irreducible provided that $c(\chi)$ is not $0, -1, \ldots,$.

Intertwining operators for $\mathrm{SL}_2(\mathbf{Q}_p)$

For simplicity take $G = \mathrm{SL}_2(\mathbf{Q}_p)$. Let $\mathfrak{x}^-$ be the negative root of $\mathfrak{g}$ relative to P.

The right translation action of $\mathfrak{x}^-$ on functions on G yields a surjective intertwining operator

$$I : \mathrm{Ind}_P^G(\chi) \to \mathrm{Ind}_P^G(\chi')$$

where $\chi' = \epsilon^{1-c(\chi)}\chi$ and ϵ is the algebraic character with $c(\epsilon) = 2$ (the positive root of of G relative to P).

Reducible principal series for $\mathrm{SL}_2(\mathbf{Q}_p)$

Suppose $G = \mathrm{SL}_2(\mathbf{Q}_p)$ and let χ be a character of the lower triangular Borel as on the previous slide. Write

$$\chi = \chi_{alg}\chi_{sm}$$

where $\chi_{alg}(\begin{matrix} t^{-1} & 0 \\ 0 & t \end{matrix}) = t^{c(\chi)}$ and χ_{sm} is smooth.

Let V_{alg} be the algebraic induction of χ_{alg} and V_{sm} the smooth induction of χ_{sm}.

There is an exact sequence of admissible locally analytic representations:

$$0 \to V_{alg} \otimes V_{sm} \to \mathrm{Ind}_P^G(K_\chi) \xrightarrow{I} \mathrm{Ind}_P^G(K_{\chi'}) \to 0$$

Remarks on Fommer-Orlik-Strauch proof of irreducibility

The proof of irreducibility rests ultimately on the Frommer-Kohlhaase theorem on the hyperenveloping algebra and on various compatibility results about uniformity.

For simplicity take $\mathbf{G} = \mathrm{GL}_n$ and $\mathbf{P}$ a Borel subgroup. Also take ρ to be a character of the maximal torus of $\mathbf{P}$ lifted back to $\mathbf{P}$.

Restriction of functions from $\mathrm{GL}_n(L)$ to $G = \mathrm{GL}_n(o_L)$ gives an isomorphism

$$\mathrm{Ind}_{\mathbf{P}(L)}^{\mathbf{G}(L)}(V) \to \mathrm{Ind}_P^G(\rho)$$

where $P = \mathbf{P}(o_L)$.

FOS theorem cont'd

It suffices to show irreducibility of $\mathrm{Ind}_P^G(\rho)$ as G-module.

Let B be an Iwahori group in G congruent to P mod πo_L.

Let $P_w = B \cap wPw^{-1}$ for w in the Weyl group and ρ^w the representation ρ conjugated by w. By the Bruhat decomposition

$$\mathrm{Ind}_P^G(\rho) = \oplus_{w\in W}\mathrm{Ind}_{P_w}^B(\rho^w).$$

FOS theorem cont'd

The associated co-admissible modules are:

$$M^w(\rho) = (\mathrm{Ind}_{P_w}^B(\rho^w))'_b.$$

One must show that these are simple $D(B, K)$ modules and that they are pairwise non-isomorphic.

For purposes of this discussion take $w = 1$.

By the Iwahori decomposition, $B = U^- P$ and U^- is the intersection of the unipotent radical of the parabolic opposite to $\mathbf{P}$ with B.

Restriction of functions to U^- identifies $\mathrm{Ind}_P^B(\chi)$ with $C^{an}(U^-, K)$ and therefore M with $D(U^-, K)$.

FOS theorem cont'd

Theorem 29. *(FOS) One can find compact open L-analytic subgroups $B_0 \subset B$, $P_0 \subset P$ and $U_0^- \subset U^-$ having uniform* structures that are compatible with the product decompositions $B = U^- P$ and $B_0 = U_0^- P_0$. As a result, the rings $D(*, K)$, where $*$ is any of the groups listed here, have compatible Fréchet-Stein structures.*

FOS theorem cont'd

Some consequences of this result are:

- $D_r(U^-, K)$ is an integral domain for r sufficiently close to 1.
- The hypotheses of the Frommer-Kohlhaase theorem are satisfied for $U_0^- \subset U^-$, and so, for $1/p < r < 1$, $D_r(U^-, K)$ is finite and free over $U(\mathfrak{u}^-, K)$ (the completion of $U(\mathfrak{u}^-) \otimes K$ in $D(U_0^-, K)$.)
- The following two families of completions of M coincide:
  - The family $M_r = D_r(B, K) \otimes M$ and
  - The family coming from identifying M with $D(U^-, K)$ and using the associated $D_r(U^-, K)$.

FOS theorem cont'd

To complete the argument, note that the algebraic Verma module

$$m(\rho) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} d\rho \xrightarrow{\sim} U(\mathfrak{u}^-)$$

in a way which is compatible with

$$m(\rho) \subset U_r(\mathfrak{u}^-, K) \subset M_r = D(U^-, K).$$

Suppose N is a $D_r(B, K)$ submodule of N.

- N is then a left ideal in $D(U^-, K)$ stable by $U(\mathfrak{t})$, where $\mathfrak{t}$ is the Lie algebra of the torus in P.
- $D_r(U^-, K)$ is a noetherian integral domain finite and free over $U_r(\mathfrak{u}^-, K)$. This implies that $N \cap U_r(\mathfrak{u}^-, K)$ is non-zero. (For $f \in N$ consider $(f, f^2, f^3, \ldots)$; noetherian property implies a relation over $U_r(\mathfrak{u}^-, K)$).
- $U_r(\mathfrak{u}^-, K)$ satisfies the "diagonalizability criterion" meaning that any non-zero $U(\mathfrak{g})$-invariant subspace of $U_r(\mathfrak{u}^-, K)$ is spanned by weight vectors coming from $m(\rho)$.

If $m(\rho)$ is simple, we see that $N = M_r$.

FOS theorem cont'd

The simplicity of M_r as $D_r(B, K)$ module for a sequence of r approaching 1 implies the simplicity of M as $D(B, K)$ module.

16 Emerton's Jacquet Modules

The smooth case

Let

- $\mathbf{G}$ be a connected reductive algebraic group over L and $G = \mathbf{G}(L)$
- $\mathbf{P}$ be a parabolic subgroup and $P = \mathbf{P}(L)$
- $\mathbf{P}^-$ be an opposite parabolic and $P^- = \mathbf{P}^-(L)$
- $\mathbf{N}$ the unipotent radical of $\mathbf{P}$, and $N = \mathbf{N}(L)$
- $\mathbf{M}$ a Levi factor of $\mathbf{P}$ and $M = \mathbf{M}(L)$
- $\mathbf{Z_G}$ and $\mathbf{Z_M}$ are the centers of $\mathbf{G}$ and $\mathbf{M}$, and Z_G and Z_M are their L-points

If V is a smooth representation of G, then the Jacquet module of V is the space V_N of N-coinvariants.

The smooth case cont'd

The Jacquet module in the smooth case has the following properties:

1. It is exact;
2. It preserves admissibility;
3. It detects parabolically induced sub- and quotient- representations.
    - If V is a smooth G-representation and W is a smooth M-representation then
$$\mathrm{Hom}_G(V, (\mathrm{Ind}_P^G(W))_{sm}) \xrightarrow{\sim} \mathrm{Hom}_M(V_N, W)$$

- If V is an admissible G-representation, W is an admissible M-representation, and δ is the modulus character of P, then

$$\mathrm{Hom}_G((\mathrm{Ind}_{P^-}^G(W))_{sm}, V) \xrightarrow{\sim} \mathrm{Hom}_M(U(\delta), V_N)$$

Essential Admissibility

Given a topologically finitely generated, locally L-analytic abelian group Z, one can construct a rigid analytic variety $\hat{Z}$ whose points K-points parameterize locally analytic characters

$$\chi : Z \to K^*$$

Indeed $Z = Z_0 \otimes \Lambda$ with Z_0 compact and Λ a free abelian group. Z_0 has a finite index subgroup isomorphic to o_L^n, and $\hat{\Lambda}$ is $\mathrm{Hom}(\Lambda, \mathbb{G}_m)$ which is representable by $dim(\Lambda)$ copies of $\mathbb{G}_m$.

The ring of entire functions $C^{an}(\hat{Z}, K)$ is a Fréchet-Stein algebra and evaluation of distributions gives an embedding $D(Z, K) \hookrightarrow C^{an}(\hat{Z}, K)$ with dense image.

Essential admissiblity cont'd

One can put a Fréchet-Stein structure on the tensor product algebra $C^{an}(\hat{Z}, K)\hat{\otimes}D(H, K)$ when H is a compact open subgroup of G.

After Emerton, a locally L-analytic representation V of G is called *essentially admissible* if

- The action of the center Z of G extends to a separately continuous $C^{an}(\hat{Z}, K)$ module structure on V.
- V_b' is a coadmissible module for $C^{an}(\hat{Z}, K)\hat{\otimes}D(H, K)$ with some (any) compact open H.

Theorem 30. *(Emerton) An admissible representation is essentially admissible.*

Emerton's Jacquet module

Suppose that V is a locally L-analytic representation of G. Let N_0 be a compact open subgroup of the unipotent group N.

Let Z_M^+ be the submonoid of Z_M consisting of elements λ such that $\lambda N_0 \lambda^{-1} \subset N_0$.

We have a "Hecke algebra" action of Z_M^+ as continuous endomorphisms of V^{N_0} consisting of operators

$$\pi_z(v) = \int_{N_0} nzv\, dn.$$

We define

$$J_P(V) = \mathcal{L}_{Z_M^+}(C^{an}(\widehat{Z_M}, K), V^{N_0}))$$

$J_P(V)$ is the "finite slope part" of V^{N_0}.

Basic properties of Emerton's functor

Emerton establishes the following properties of $J_P(V)$ (point (iii) in particular is a serious theorem):

1. $J_P(V)$ is independent (up to natural isomorphism) of the choices of Levi factor and compact open subgroup N_0 of N used to define it.
2. $J_P(V)$ is a locally analytic representation of M.
3. $J_P(V)$ is left-exact.
4. The functor $J_P(V)$ (viewed as a functor from P-representations to M-representations) is right adjoint to the functor which associates to a locally analytic representation of M, on a vector space U of compact type, where the Z_M action extends to $C^{an}(\widehat{Z}_M, K)$ action, the P representation $C_c(N, U)$ of compactly supported U-valued locally constant functions on N.
5. $J_P(V)$ takes essentially admissible representations of G to essentially admissible representations of M.

Emerton's Functor and Banach-Hecke algebras

Suppose for simplicity that G is split, that P is a Borel subgroup, and M is a torus.

In Dat's lectures on Banach-Hecke algebras we saw that one could find that the existence of a G-invariant norm on a locally algebraic representation of the form $W_\chi \otimes X$, with W_χ the algebraic representation with highest weight χ and X smooth, forced a certain condition on $d\chi$.

In terms of Jacquet modules, Emerton computes

$$J_P(W(\chi) \otimes X) = W^N \otimes X_N$$

Note that W^N is the representation χ.

Emerton's functor and Banach-Hecke algebras cont'd

Theorem 31. *Suppose that V is a locally algebraic representation $W \otimes X$ of G that admits a P-invariant norm. Then if the χ-eigenspace of $J_P(V)$ is non-zero, we must have $|\delta(a)^{-1}\chi(a)| \leq 1$ for every a in $Z_M^+ = M^+$.*

Once needs to know the left exactness of J_P, the formula for J_P of a locally algebraic representation as above, and the fact that the χ-eigenspace of $J_P(V)$ coincides with that of V^{N_0} – that is, that the finite slope part is harmless on χ-eigenspaces. Then this result follows by looking at the explicit formula for the action of the Hecke operators as sums much as in the Banach-Hecke case.

Taking the Weyl group into account and using the left exactness of J_P, we see that this theorem implies the same condition on $d\chi$ as arises in the Banach-Hecke context.

Part III
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