(φ,Γ) - modules and the p-adic Langlands correspondence

Part I : Laurent Berger - notes provided "as is"

(φ,Γ) - modules
p - adic Hodge theory

In this course, I will:
- give definitions
- provide an overview of the relevant results

I will NOT:
- prove the results
- motivate the constructions
- give the most general definitions

Notations: K is a finite extension of $\mathbb{Q}_p$, usually $K = \mathbb{Q}_p$

$G_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p / K)$

$\zeta_{p^n} = p^n$-th root of 1 $\quad (\zeta_{p^{n+1}}^p = \zeta_{p^n})$

$K_n = K(\zeta_{p^n}) \qquad K_\infty = \bigcup_{n \geq 1} K_n$

$H_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p / K_\infty)$

$\Gamma = \mathrm{Gal}(K_\infty / K) \xrightarrow{\chi} \mathbb{Z}_p^\times$

V is a rep$^{\underline{n}}$ of G_K over $\mathbb{Q}_p$

T is a $\mathbb{Z}_p$-rep$^{\underline{n}}$ and W is a $\mathbb{F}_p$-rep$^{\underline{n}}$

(ex $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T \qquad W = \mathbb{F}_p \otimes_{\mathbb{Z}_p} T$)

Contents:

1. $\mathbb{F}_p$-rep$^{\underline{ns}}$ and (φ,Γ)-modules
2. (φ,Γ)-modules in characteristic 0
3. Slopes of φ-modules over $\mathcal{R}$
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5. Construction of (φ,Γ)-modules
6. (φ,Γ)-modules and p-adic rep$^{\underline{ns}}$
7. p-adic Hodge theory
8. Wach modules
9. The operator ψ
10. The mod p correspondence

① $\mathbb{F}_p$-rep$^{\text{ns}}$ of G_K and (φ,Γ)-modules

We want to describe $\mathbb{F}_p$-reps of G_K

Let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$ $\quad \mathcal{O}_{\mathbb{C}_p}$ = integers of $\mathbb{C}_p$ $\quad \widetilde{\mathbb{E}}^+ = \varprojlim \mathcal{O}_{\mathbb{C}_p}/p$

If $y = (y_0, y_1, \dots) \in \widetilde{\mathbb{E}}^+$ $\quad v_E(y) = \lim p^n v_p(\hat{y}_n)$ = valuation on $\widetilde{\mathbb{E}}^+$

and $\widetilde{\mathbb{E}}^+$ is complete for v_E

Let $\varepsilon = (1, \overline{\zeta}_p, \overline{\zeta}_{p^2}, \dots)$ and $X = \varepsilon - 1$ $\quad v_E(X) = \frac{p}{p-1}$

$\widetilde{\mathbb{E}} = \widetilde{\mathbb{E}}^+[\frac{1}{X}]$ = algebraically closed field of char. p

note: $\mathbb{F}_p((X)) \subset \widetilde{\mathbb{E}}$ and $\mathbb{E} = \mathbb{F}_p((X))^{sep}$ is dense in $\widetilde{\mathbb{E}}$

frobenius: φ on $\widetilde{\mathbb{E}}$ $\quad y \mapsto y^p$ $\quad \widetilde{\mathbb{E}}^{\varphi=1} = \mathbb{F}_p$

Galois: $G_{\mathbb{Q}_p}$ acts on $\widetilde{\mathbb{E}}$ $\quad g(\varepsilon - 1) = \varepsilon^{\chi(g)} - 1 = (1+X)^{\chi(g)} - 1$

so that $H_{\mathbb{Q}_p}$ acts trivially on $\mathbb{F}_p((X))$ so we get a map:

$$H_{\mathbb{Q}_p} \longrightarrow \mathrm{Gal}(\mathbb{E} / \mathbb{F}_p((X)))$$

this map is: • injective - if h acts by id on $\mathbb{E}$ $\Rightarrow$ also on $\widetilde{\mathbb{E}}$ $\Rightarrow$ also on $\mathbb{C}_p$ $\Rightarrow$ $h = id$

• surjective - if $g: \mathbb{E} \to \mathbb{E}$ $\rightsquigarrow$ $g: \widetilde{\mathbb{E}} \to \widetilde{\mathbb{E}}$ and because $g(\varepsilon) = \varepsilon$ (since g is trivial on $\mathbb{F}_p((X))$), $[g(x_0, x_1, \dots)]_0$ only depends on x_0 $\Rightarrow$ g comes from a map $\mathbb{C}_p \to \mathbb{C}_p$

Hilbert 90 tells us that $H^1_{\text{disc.}}(H_{\mathbb{Q}_p}, GL_d(\mathbb{E})) = \{1\}$

if W is a $\mathbb{F}_p$-rep$^{\text{n}}$ of $G_{\mathbb{Q}_p}$ $\Rightarrow$ $[W] \in H^1(H_{\mathbb{Q}_p}, GL_d(\mathbb{F}_p))$

and since $[W] \mapsto \{1\}$ in $H^1(H_{\mathbb{Q}_p}, GL_d(\mathbb{E}))$, we get:

$$\mathbb{E} \otimes_{\mathbb{F}_p} W \simeq \mathbb{E}^d \text{ as reps of } H_{\mathbb{Q}_p}$$

so that $\mathbb{D}(W) = (\mathbb{E} \otimes W)^{H_{\mathbb{Q}_p}}$ is a $\mathbb{F}_p((X))$-vector space of dimension d ($= \dim_{\mathbb{F}_p} W$), + frob. φ and an action of Γ

A (φ,Γ)-module over $\mathbb{F}_p((X))$ is a $\mathbb{F}_p((X))$-vec. space of finite dimension + φ such that $\mathrm{Mat}(\varphi) \in GL_d(\mathbb{F}_p((X)))$ and a compatible action of Γ.

In particular, $\mathbb{D}(W)$ is a (φ,Γ)-module over $\mathbb{F}_p((X))$

In addition, $\mathbb{E}\otimes\mathbb{D}(W) = \mathbb{E}\otimes W \Rightarrow W = (\mathbb{E}\otimes\mathbb{D}(W))^{\varphi=1}$

Finally if $\mathbb{D}$ is a (φ,Γ)-module over $\mathbb{F}_p((X))$, let:

$$W = (\mathbb{E}\otimes\mathbb{D})^{\varphi=1}$$

If $\mathrm{Mat}(\varphi) = (q_{ij})$ then $\varphi(A)\cdot\mathrm{Mat}(\varphi) = A$ becomes:

$\sum_j a_{ij}^p q_{jk} = a_{ik}$ and $\mathbb{E}[a_{ij}]/(\sum_j a_{ij}^p q_{jk} - a_{ik})$ is an étale algebra since $\mathrm{Mat}(\varphi)\in GL_d(\mathbb{F}_p((X)))$ so card $W = p^d$ so that W is a $\mathbb{F}_p$-repn of dimension d.

Theorem: The functor $W \mapsto \mathbb{D}(W)$

$$\{\mathbb{F}_p\text{-rep}^{\text{ns}} \text{ of } G_{\mathbb{Q}_p}\} \longrightarrow \{(\varphi,\Gamma)\text{-modules over } \mathbb{F}_p((X))\}$$

is an equivalence of categories

example: let ω_n be Serre's fundamental character of level n and $h\in\mathbb{Z}$ and $\mathrm{ind}(\omega_n^h)$ = the n-diml $\mathbb{F}_p$-repn of $G_{\mathbb{Q}_p}$ whose det is ω^h (ω = mod p cyclo. char.) and whose restriction to inertia is $\omega_n^h\oplus\omega_n^{ph}\oplus\cdots\oplus\omega_n^{p^{n-1}h}$. Then in a good basis of $\mathbb{D}(\mathrm{ind}(\omega_n^h))$ we have:

$$\mathrm{Mat}(\varphi) = \begin{pmatrix} 0 & & & \pm X^{-h(p-1)} \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

$$\mathrm{Mat}(\gamma) = \begin{pmatrix} \lambda_\gamma^{h\cdot\frac{p-1}{p^n-1}} & & \\ & \ddots & \\ & & \lambda_\gamma^{h\cdot p^{n-1}\frac{p-1}{p^n-1}} \end{pmatrix}$$

where $\lambda_\gamma = \dfrac{\omega(\gamma)X}{\gamma(X)} \in 1 + X\cdot\mathbb{F}_p[[X]]$

② (φ,Γ)-modules in char. 0

Choose $\rho<1$ with $\rho\in p^{\mathbb{Q}}$ (i.e. $\rho\in|\mathfrak{m}_{\mathbb{C}_p}|_p$)

Let $\mathcal{R}^\rho=\{f(X)=\sum_{k\in\mathbb{Z}} a_k X^k \quad a_k\in\mathbb{Q}_p,$
$\qquad f(X)$ converges on $\rho\leq|X|_p<1\}$

$\mathcal{R}=\bigcup_{\rho<1}\mathcal{R}^\rho$ = the Robba ring

$\mathcal{E}^t=\{f(X)\in\mathcal{R}$ with bounded coefficients $\}$

$\mathcal{O}_{\mathcal{E}}^t=\{f(X)\in\mathcal{R},\ |a_k|\leq 1\ \forall k\}$

$\mathcal{E}^t$ is a field and $\mathcal{R}^\times=\mathcal{E}^t\setminus\{0\}$

If I is a closed interval $\subset[\rho;1[$, we have a $\begin{cases}\text{valuation } V_I\\ \text{norm } |\cdot|_I\end{cases}$ on $\mathcal{R}^\rho$: $|f(X)|_I=\sup_{|z|\in I}|f(z)|_p$

$\rightsquigarrow$ Fréchet topology on $\mathcal{R}^\rho$.

If $\mathcal{E}^{t,\rho}=\mathcal{E}^t\cap\mathcal{R}^\rho$ then $\mathcal{E}^{t,\rho}$ is dense in $\mathcal{R}^\rho$

Each $\mathcal{E}^{t,\rho}$ is a p.i.d and $\mathcal{R}^\rho$ is a Bézout ring.

More generally, if M is a submodule of $(\mathcal{R}^\rho)^d$ for some $d\geq 1$, then M is free $\Leftrightarrow$ M is closed $\Leftrightarrow$ M is of finite type.

On $\mathcal{E}^t$ there is also the Gauss norm $|f(X)|_G=\sup_k |a_k|$ and the completion of $\mathcal{E}^t$ for $|\cdot|_G$ is $\mathcal{E}=\{f(X)=\sum_{k\in\mathbb{Z}}a_kX^k$, a_k bounded and $a_k\xrightarrow[k\to-\infty]{}0\}$. This is a 2-dim$^{\underline{l}}$ local field, whose ring of integers is $\mathcal{O}_{\mathcal{E}}$ and we have $\mathcal{O}_{\mathcal{E}}/p=\mathbb{F}_p((X))$.

On $\mathcal{R}$ and $\mathcal{E}^t$ and $\mathcal{E}$ we have:

a frobenius $\varphi: f(X)\mapsto f((1+X)^p-1)$

an action of Γ, γ: $f(X)\mapsto f((1+X)^{\chi(\gamma)}-1)$

Let $\mathcal{A}$ be $\mathcal{R}$ or $\mathcal{E}^t$ or $\mathcal{E}$.

A φ-module on $\mathcal{A}$ is a free $\mathcal{A}$-module of finite rank + a semilinear φ such that $\mathrm{Mat}(\varphi)\in GL_d(\mathcal{A})$.

A (φ,Γ)-module is a φ-module + a compatible action of Γ.
note: $\mathcal{E}^{t}$ and $\mathcal{E}$ are fields so in these two cases, (φ,Γ)-modules are vector spaces.

③ Slopes of φ-modules over $\mathcal{R}$

If $\mathbb{D}$ is a φ-module of rank 1 over $\mathcal{R}$ then $\mathrm{Mat}(\varphi) \in \mathcal{R}^{\times}$ and so $\exists!\, n \in \mathbb{Z}$ $\mathrm{Mat}(\varphi) \in p^{n} \cdot (\mathcal{O}_{\mathcal{E}}^{t})^{\times}$. n is the slope of φ on $\mathbb{D}$, we set $\deg(\mathbb{D}) = n$. If $\mathbb{D}$ is of rank ≥ 1, we set $\deg(\mathbb{D}) = \deg \det(\mathbb{D})$.

Let $s = n/a \in \mathbb{Q}$ $(a \geq 1)$. We say a φ-module $\mathbb{D}$ is isoclinic of slope s if $\exists$ a basis of $\mathbb{D}$ in which:

$$\mathrm{Mat}(p^{-n}\varphi^{a}) \in GL_d(\mathcal{O}_{\mathcal{E}}^{t})$$

This is well defined since slope = deg / dim.

If $s=0$, we say $\mathbb{D}$ is étale.

theorem: If $\mathbb{D}$ is a φ-module over $\mathcal{R}$, then $\exists$ a canonical filtration $0 = \mathbb{D}_0 \subset \mathbb{D}_1 \subset \cdots \subset \mathbb{D}_\ell = \mathbb{D}$ such that each $\mathbb{D}_i / \mathbb{D}_{i-1}$ is isoclinic, and the slopes of the $\mathbb{D}_i/\mathbb{D}_{i-1}$ are increasing.

note:
- the filtration is not split in general
- the filtration is canonical $\Rightarrow$ if $\mathbb{D}$ is a (φ,Γ)-module then so is each $\mathbb{D}_i$.

The slopes of $\mathbb{D}$ are the slopes of the $\mathbb{D}_i/\mathbb{D}_{i-1}$ with multiplicity $\mathrm{rk}(\mathbb{D}_i/\mathbb{D}_{i-1})$.

example: If M is a φ-module over $\mathbb{Q}_p$ then by Dieudonné–Manin, $\widehat{\mathbb{Q}}_p^{nr} \otimes M = \oplus M_{[r]}$ $M_{[r]}$ is isoclinic of slope r and $\mathbb{D} = \mathcal{R} \otimes M$ is a φ-module over $\mathcal{R}$ whose slopes are those of ~~$\mathcal{M}$~~ M.

If D^t is a (φ,Γ)-module over $\mathcal{E}^t$, we say it's étale if $\exists$ a basis in which $\mathrm{Mat}(\varphi) \in GL_d(\mathcal{O}_\mathcal{E}^t)$ so that $\mathbb{D} = \mathcal{R} \otimes D^t$ is étale. The functor:

$$\{\text{étale } \varphi\text{-modules } / \mathcal{E}^t\} \to \{\text{étale } \varphi\text{-modules } / \mathcal{R}\}$$

is then an equivalence of categories.

④ Extensions of (φ,Γ)-modules

The slope filtration theorem shows that (φ,Γ)-modules are iterated extensions of isoclinic objects. Let $\mathbb{D}$ be a (φ,Γ)-module and assume that $\Gamma = \overline{\langle \gamma \rangle}$ (true if $\Gamma \neq \mathbb{Z}_2^\times$).

Consider the complex:

$$C^\bullet_\varphi(\mathbb{D}) : \quad 0 \to \mathbb{D} \to \mathbb{D} \oplus \mathbb{D} \to \mathbb{D} \to 0$$

$$x \mapsto ((\varphi-1)x, (\gamma-1)x)$$

$$(a,b) \mapsto (\gamma-1)a - (\varphi-1)b$$

$$H^0(C^\bullet_\varphi(\mathbb{D})) = \mathbb{D}^{\varphi=1,\gamma=1} = \mathrm{Hom}(\mathcal{R},\mathbb{D})$$

Theorem:
- $H^i(C^\bullet_\varphi(\mathbb{D})) \simeq \mathrm{Ext}^i(\mathcal{R},\mathbb{D})$
- $\mathrm{Ext}^i(\mathcal{R},\mathbb{D})$ is a finite dim^{l} $\mathbb{Q}_p$-vector space, $= 0$ if $i \geq 3$
- $\dim \mathrm{Ext}^0 - \dim \mathrm{Ext}^1 + \dim \mathrm{Ext}^2 = -\mathrm{rk}(\mathbb{D})$
- there is a perfect pairing $\mathrm{Ext}^i(\mathcal{R},\mathbb{D}) \times \mathrm{Ext}^{2-i}(\mathcal{R},\mathbb{D}) \to \mathrm{Ext}^2(\mathcal{R},\mathbb{D}) = \mathbb{Q}_p$
- and if we define $C^\bullet_\varphi(D^t)$ for D^t over $\mathcal{E}^t$ then $C^\bullet_\varphi(D^t)$ and $C^\bullet_\varphi(\mathcal{R} \otimes D^t)$ are quasi-isomorphic

(φ,Γ)-modules of rank 1: if $\delta : \mathbb{Q}_p^\times \to \mathbb{Q}_p^\times$ is a continuous character, we define $\mathcal{R}(\delta) = \mathcal{R}\cdot e$ where,

$$\varphi e = \delta(p)\cdot e \qquad (\text{slope of } \varphi = v_p(\delta(p)))$$

$$\gamma e = \delta(\chi(\gamma))\cdot e$$

and every (φ,Γ)-module of rank 1 arises in this way.

We say that a (φ,Γ)-module is trianguline if it is an iterated extension of (φ,Γ)-modules of rank 1 (after possibly extending scalars from $\mathbb{Q}_p$ to a finite extension L)

In dimension 2, such $\mathbb{D}$'s arise as:

$$0 \to \mathcal{R}(\delta_1) \to \mathbb{D} \to \mathcal{R}(\delta_2) \to 0$$

and so are classified by $\mathrm{Ext}^1(\mathcal{R}(\delta_2), \mathcal{R}(\delta_1))$

5 Construction of (φ,Γ)-modules

Recall that
- $\mathcal{R}^\rho$ = functions on $\{\rho \leq |X|_p < 1\}$
- ζ_{p^n} = p^n-th root of 1
- $K_n = \mathbb{Q}_p(\zeta_{p^n})$

We have $|\zeta_{p^n} - 1|_p = p^{-\frac{1}{p^{n-1}(p-1)}} \xrightarrow[n\to+\infty]{} 1 \Rightarrow |\zeta_{p^n}-1|_p \geq \rho$ if $n \geq n(\rho)$

If $n \geq n(\rho)$ we then have a map:

$$\iota_n : \mathcal{R}^\rho \longrightarrow K_n[[t]]$$
$$f(X) \longmapsto f(\zeta_{p^n}\exp(t/p^n) - 1)$$

which is Γ-equivariant ($\gamma(t) = \chi(\gamma)t$).

If we also denote by t the series $\log(1+X) \in \mathcal{R}^\rho$, then $\iota_n(t) = p^{-n}t$.

Note that $\iota_{n+1}\circ\varphi = \iota_n$.

We add a variable ℓ_X to $\mathcal{R}$ and we define:

$$\varphi(\ell_X) = p\,\ell_X + \log\left(\frac{\varphi X}{X^p}\right)$$
$$\gamma(\ell_X) = \ell_X + \log\left(\frac{\gamma X}{X}\right)$$

so that ℓ_X is really $\log(X)$.

We extend ι_n to $\mathcal{R}^\rho[\log X]$ by:

$$\iota_n : \log X \mapsto \log(\zeta_{p^n}\exp(t/p^n) - 1)$$

to do this we need to choose $\log p$ and we choose $\log p = 0$.

Finally we get by inverting t a map:

$$\iota_n : \mathcal{R}^{+}[\log X, \tfrac{1}{t}] \longrightarrow K_n((t))$$

which is injective.

A filtered (φ, N)-module is a finite dim$^{\underline{l}}$ $\mathbb{Q}_p$-vector space D plus a linear φ and N such that $N\varphi = p\varphi N$ (in particular, N is nilpotent), and a filtration $\mathrm{Fil}^i D$.

Using the ι_n maps, we can construct (φ,Γ)-modules from filtered (φ,N)-modules. Define N on $\mathcal{R}[\log X]$ by $N = -\frac{p}{p-1}\frac{d}{d\log X}$

If D is a ~~filte~~ filtered (φ,N)-module, let:

$$\mathbb{D} = (\mathcal{R}[\log X] \otimes D)^{N=0}$$

which is a (φ,Γ)-module $/\mathcal{R}$ of rank $= \dim D$, such that $\mathcal{R}[\log X] \otimes \mathbb{D} = \mathcal{R}[\log X] \otimes D$ and let:

$$\mathcal{M}(D) = \{ y \in \mathcal{R}[\log X, \tfrac{1}{t}] \otimes D \text{ s.t.:}$$

- $Ny = 0$ (in particular $y \in \mathcal{R}[\frac{1}{t}] \otimes \mathbb{D}$)
- $\forall n \geq n(y)$, $\iota_n(y) \in \mathrm{Fil}^0 (K_n((t)) \otimes D)$ $\}$

Note that if $\mathrm{Fil}^{b+1} D = 0 \Rightarrow \mathcal{M}(D) \subset t^{-b}\,\mathbb{D}$

$\mathrm{Fil}^{a} D = D \Rightarrow t^{-a}\mathbb{D} \subset \mathcal{M}(D)$

In addition, ~~as~~ N and each ι_n is continuous $\Rightarrow \mathcal{M}(D)$ is a closed submodule of $t^{-b}\,\mathbb{D}$ which contains $t^{-a}\,\mathbb{D}$ $\Rightarrow$

$\mathcal{M}(D)$ is a (φ,Γ)-module $/\mathcal{R}$ of rank $= \dim D$.

Note also that $\mathcal{R}[\frac{1}{t}] \otimes \mathcal{M}(D) = \mathcal{R}[\frac{1}{t}] \otimes \mathbb{D}$ so that:

$$\mathcal{R}[\log X, \tfrac{1}{t}] \otimes \mathcal{M}(D) = \mathcal{R}[\log X, \tfrac{1}{t}] \otimes D.$$

We say that a (φ,Γ)-module $\mathbb{M}$ is unipotent if

$$\dim (\mathcal{R}[\log X, \tfrac{1}{t}] \otimes \mathbb{M})^{\Gamma} = \mathrm{rk}(\mathbb{M})$$

We therefore get a functor:

$$D \longmapsto \mathcal{M}(D)$$

$$\{\text{filtered } (\varphi,N)\text{-modules}\} \to \{\text{unipotent } (\varphi,\Gamma)\text{-modules } / \mathcal{R}\}$$

which is an equivalence of categories.

If D is of dim. 1 then $\mathrm{Mat}(\varphi) = \pi \in \mathbb{Q}_p^\times$ and we set $t_N(D) = v_p(\pi)$.
Also $\exists!\, h$ such that $\mathrm{Fil}^h D = D$ and $\mathrm{Fil}^{h+1} D = 0$ and $t_H(D) = h$.
If D is of dim. ≥ 1, set $t_N(D) = t_N(\det D)$ and $t_H(D) = t_H(\det D)$.
We say that D is admissible if:

- $t_H(D) = t_N(D)$
- for every $D' \subset D$, $t_N(D') \geq t_H(D')$.

If $\dim(D) = 1$ and $h = t_H(D)$ and $n = t_N(D)$, then $D = \mathbb{Q}_p \cdot e$ with $\varphi e = p^n \pi_0 e$ $(\pi_0 \in \mathbb{Z}_p^\times)$ and $\mathcal{M}(D) = \mathcal{R}\cdot t^{-h} e$ and $\varphi(t^{-h}e) = p^{n-h}\pi_0 \cdot t^{-h} e \Rightarrow \deg \mathcal{M}(D) = t_N(D) - t_H(D)$.

Theorem: The filtered (φ,N)-module is ~~étale~~ admissible $\iff$ $\mathcal{M}(D)$ is étale.

Indeed, $t_H(D) = t_N(D) \iff \deg \mathcal{M}(D) = 0$
and the saturated subobjects of $\mathcal{M}(D)$ are of the form $\mathcal{M}(D')$
so D admissible $\iff$ all subobjects of $\mathcal{M}(D)$ are of degree ≥ 0.

6 (φ,Γ)-modules and p-adic representations.

To relate (φ,Γ)-modules and p-adic rep$^{\text{ns}}$, we start with (φ,Γ)-modules over $\mathcal{E}$. Such an object is étale if $\exists$ a basis in which $\mathrm{Mat}(\varphi) \in GL_d(\mathcal{O}_\mathcal{E})$.
Recall that we defined $\tilde{\mathbb{E}}^+ = \varprojlim \mathcal{O}_{\mathbb{C}_p}$ and $\tilde{\mathbb{E}} = \tilde{\mathbb{E}}^+[\frac{1}{x}]$.
Let $\tilde{\mathbb{A}} = W(\tilde{\mathbb{E}}) = \{\sum_{k\geq 0} p^k [x_k],\ x_k \in \tilde{\mathbb{E}}\}$ and $\tilde{\mathbb{B}} = \tilde{\mathbb{A}}[\frac{1}{p}]$.

There is a frobenius φ on $\tilde{\mathbb{B}}$, such that $\tilde{\mathbb{B}}^{\varphi=1} = \mathbb{Q}_p$ and an action of $G_{\mathbb{Q}_p}$.

Set $X = [\varepsilon]-1 \in \tilde{\mathbb{A}}$ so that $X \overset{\text{mod } p}{\mapsto} X \in \tilde{\mathbb{E}}$

let $\mathcal{O}_\mathcal{E}$ = p-adic completion of $\mathbb{Z}_p[[X]][\frac{1}{X}] \subset \tilde{\mathbb{A}}$

$\mathcal{E} = \mathcal{O}_\mathcal{E}[\frac{1}{p}] \subset \tilde{\mathbb{B}}$

let $\mathcal{E}^{nr}$ be the maximal unramified extension of $\mathcal{E}$ inside $\tilde{\mathbb{B}}$, $\widehat{\mathcal{E}^{nr}}$ its p-adic completion (this ring is also denoted by $\mathbb{B}$)

$Aut(\mathbb{B}/\mathcal{E}) = Gal(\mathcal{E}^{nr}/\mathcal{E}) = Gal(\mathbb{E}/\mathbb{F}_p((X))) = H_{\mathbb{Q}_p}$

If T is a $\mathbb{Z}_p$-rep of $G_{\mathbb{Q}_p}$, we set $\mathbb{D}(T) = (\mathcal{O}_{\widehat{\mathcal{E}^{nr}}} \otimes T)^{H_{\mathbb{Q}_p}}$

- if T is p-torsion then ~~the~~ this is the construction from §1
- if T is p^n-torsion then we prove by induction on n that $\mathcal{O}_{\widehat{\mathcal{E}^{nr}}} \otimes \mathbb{D}(T) = \mathcal{O}_{\widehat{\mathcal{E}^{nr}}} \otimes T$

and by passing to the limit we get:

theorem: | The functor $T \mapsto \mathbb{D}(T)$
$\{\mathbb{Z}_p\text{-reps of } G_{\mathbb{Q}_p}\} \to \{(\varphi,\Gamma)\text{-modules over } \mathcal{O}_\mathcal{E}\}$
is an equivalence of categories.

by inverting p, we get an equivalence of categories:

$$\{p\text{-adic reps of } G_{\mathbb{Q}_p}\} \to \{\text{étale } (\varphi,\Gamma)\text{-modules } / \mathcal{E}\}$$

In particular, if $\mathbb{D}^t$ is an étale (φ,Γ)-module over $\mathcal{E}^t$ then $\mathcal{E} \otimes \mathbb{D}^t$ is étale / $\mathcal{E}$ $\rightsquigarrow$ p-adic repⁿ, so we get a functor:

$$\{\text{étale } (\varphi,\Gamma)\text{-modules } / \mathcal{E}^t\} \longrightarrow \{p\text{-adic reps of } G_{\mathbb{Q}_p}\}$$

theorem: | This functor is also an equivalence of categories

The proof is quite hard, it uses Colmez' "decompletion" techniques.

As a corollary, we get:

$$\begin{array}{ccc} & \{\text{étale } (\varphi,\Gamma)\text{-modules} / \mathcal{E}^t\} & \\ /\!/ & & \backslash\!\backslash \\ \{\text{étale } (\varphi,\Gamma)\text{-modules} / \mathcal{R}\} & = & \{\text{étale } (\varphi,\Gamma)\text{-modules} / \mathcal{E}\} \end{array}$$

although $\mathcal{E}$ and $\mathcal{R}$ are "unrelated".

Recall that in §4 we constructed the complex $C^\bullet_\varphi(\mathbb{D}^t)$ whose cohomology computed $\mathrm{Ext}^i(\mathcal{E}^t, \mathbb{D}^t)$. Using the above equivalence of categories, we get:

Theorem: If V is a p-adic rep^{n}, then the cohomology of $C^\bullet_\varphi(\mathbb{D}^t(V))$ is the Galois cohomology of V.
In particular:
- $H^i(G_{\mathbb{Q}_p}, V) = 0$ if $i \geq 3$
- Euler-Poincaré formula
- Tate duality

⑦ p-adic Hodge theory

Recall that $\tilde{\mathbb{E}}^+ = \varprojlim \mathcal{O}_{\mathbb{C}_p}$. We set $\tilde{\mathbb{A}}^+ = W(\tilde{\mathbb{E}}^+)$ and $\tilde{\mathbb{B}}^+ = \tilde{\mathbb{A}}^+[\frac{1}{p}]$

and we define a map:

$$\theta : \tilde{\mathbb{B}}^+ \longrightarrow \mathbb{C}_p$$
$$\sum p^k [x_k] \mapsto \sum p^k x_k^{(0)}$$

where $x_k^{(n)} = \lim_j \hat{x}_{k+n+j}^{p^j}$. This defines a surjective morphism of rings,

and $\ker(\theta) = \left(\frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1}\right) \cdot \tilde{\mathbb{B}}^+$.

Define $\mathbb{B}_{dR}^+ = \varprojlim \tilde{\mathbb{B}}^+ / (\ker\theta)^n$

This ring has an action of $G_{\mathbb{Q}_p}$ but no φ ($\varphi(\ker\theta) \not\subset \ker\theta$)

If $K/\mathbb{Q}_p$ is finite then $(\mathbb{B}_{dR}^+)^{G_K} = K$ (although $(\tilde{\mathbb{B}}^+)^{G_K} = K_0$ where $K_0 = K \cap \mathbb{Q}_p^{nr}$).

Let $t = \log([\varepsilon]) = -\sum_{n\geq 1} \frac{(1-[\varepsilon])^n}{n} \in \mathbb{B}_{dR}^+$, so that $g(t) = \chi(g)t$ if $g \in G_{\mathbb{Q}_p}$. We then have $\ker\theta = t\cdot\mathbb{B}_{dR}^+$ and $\mathbb{B}_{dR}^+/t \simeq \mathbb{C}_p$.
The map $\tilde{\mathbb{B}}^+ \to \mathbb{B}_{dR}^+/t^h$ is onto and we define a norm on $\mathbb{B}_{dR}^+/t^h$ by image of $\tilde{\mathbb{A}}^+$ = unit ball. This gives $\mathbb{B}_{dR}^+ = \varprojlim \frac{\mathbb{B}_{dR}^+}{t^h}$ a Fréchet topology.

Let $\mathbb{B}_{dR} = \mathbb{B}_{dR}^+[\frac{1}{t}]$; this is a field, with:

- an action of $G_{\mathbb{Q}_p}$
- a filtration $\mathrm{Fil}^i\mathbb{B}_{dR} = t^i\,\mathbb{B}_{dR}^+$
- a LF topology

If V is a p-adic rep of $G_{\mathbb{Q}_p}$ we set $D_{dR}(V) = (\mathbb{B}_{dR}\otimes V)^{G_{\mathbb{Q}_p}}$.
This is a $\mathbb{Q}_p$-vector space of dim $\leq \dim(V)$ and we say that V is de Rham if we have equality. This is equivalent to:

$$\mathbb{B}_{dR}\otimes D_{dR}(V) \xrightarrow{\sim} \mathbb{B}_{dR}\otimes V$$

$D_{dR}(V)$ is a filtered vector space, and we get a functor:

$$\{\text{de Rham reps}\} \longrightarrow \{\text{filtered } \mathbb{Q}_p\text{-vector spaces}\}$$

which forgets a lot of information (~~one~~ one cannot recover V from $D_{dR}(V)$)

We construct some smaller rings which have more structure.
Let $\tilde{p}\in\tilde{\mathbb{E}}^+$ be such that $p^{(0)} = p$.
Define $\mathbb{A}_{max} = \tilde{\mathbb{A}}^+\{\frac{[\tilde p]}{p}\}$ $(= \tilde{\mathbb{A}}^+\{X\}/(pX - [\tilde p])$, which injects in $\mathbb{B}_{dR}^+$ by $X \mapsto [\tilde p]/p$).

$\mathbb{B}_{max}^+ = \mathbb{A}_{max}[1/p]$ this has an action of $G_{\mathbb{Q}_p}$ and a frobenius φ. Let $\mathbb{B}_{max} = \mathbb{B}_{max}^+[\frac{1}{t}]$. This is similar to $\mathbb{B}_{cris}$ but better in some ways. Finally let $\mathbb{B}_{st} = \mathbb{B}_{max}[u]$ where $\varphi u = pu$, $Nu = -1$. We have $\mathbb{B}_{st} \hookrightarrow \mathbb{B}_{dR}$ by $u \mapsto \log[\tilde p] = \underbrace{\log p}_{(=0)} + \log\frac{[\tilde p]}{p}$ $\left(\theta\left(\frac{[\tilde p]}{p}\right)=1\right)$.

We view all these rings as subrings of $\mathbb{B}_{dR}$.

If V is a p-adic rep of $G_{\mathbb{Q}_p}$ we say that V is:

- semi-stable if $\dim\left(D_{st}(V) = (\mathbb{B}_{st}\otimes V)^{G_{\mathbb{Q}_p}}\right) = \dim(V)$
- crystalline if $\dim\left(D_{cris}(V) = (\mathbb{B}_{cris}\otimes V)^{G_{\mathbb{Q}_p}}\right) = \dim(V)$

Note that $D_{st}(V)$ is a filtered (φ, N)-module (filtered via $D_{st}(V) \hookrightarrow D_{dR}(V)$) and that $D_{cris}(V) = D_{st}(V)^{N=0}$.

We now turn to the relationship between p-adic Hodge theory and (φ,Γ)-modules. There exists a "big ring" $\tilde{\mathbb{B}}^{t}_{rig}$ such that:

- $\mathcal{R} \hookrightarrow \tilde{\mathbb{B}}^{t}_{rig}$
- $\bigcap_{n\geq 0} \varphi^n(\mathbb{B}_{st}) \hookrightarrow \tilde{\mathbb{B}}^{t}_{rig}[\log X]$

So that if V is a p-adic rep^{n} then:

- $\mathbb{D}(V) \subset \tilde{\mathbb{B}}^{t}_{rig} \otimes V \subset \tilde{\mathbb{B}}^{t}_{rig}[\log X, \frac{1}{t}] \otimes V$
- $D_{st}(V) \subset \tilde{\mathbb{B}}^{t}_{rig}[\log X, \frac{1}{t}] \otimes V$

theorem: We have $D_{st}(V) = \left(\mathcal{R}[\log X, \frac{1}{t}] \otimes \mathbb{D}(V)\right)^{\Gamma}$, so that V is semi-stable $\Leftrightarrow$ $\mathbb{D}(V)$ is unipotent.

Finally, recall that in §4 we defined trianguline (φ,Γ)-modules. We say that a rep^{n} V is trianguline if $\mathbb{D}(V)$ is trianguline.

examples:
- semi-stable rep^{ns} are trianguline
- dR + trianguline $\Rightarrow$ becomes sst / an abelian ext. of $\mathbb{Q}_p$
- if V = p-adic rep^{n} assoc. to a modular form $\Rightarrow$ V is trianguline.

$$\{\text{all } p\text{-adic reps}\} \supset \{dR\} \supset \{sst\} \supset \{cris\}$$
$$\{sst\} \subset \{\text{trianguline}\}$$

⑧ Wach modules

If V is a crystalline representation, then its (φ,Γ)-module is nicer than for arbitrary representations.

Let $\mathcal{R}^+ = \{ f(x) \in \mathcal{R},\ f(x) = \sum_{k \geq 0} a_k X^k \}$

$\mathcal{E}^+ = \mathcal{R}^+ \cap \mathcal{E}^t = \mathbb{Q}_p \otimes \mathbb{Z}_p[[X]]$

Theorem: If V is crystalline, then $\mathbb{D}(V)$ has a basis in which $\mathrm{Mat}(\varphi) \in M_d(\mathcal{E}^+)$.

Actually, we have more precise results. A Wach module is a free $\mathcal{E}^+$-module $\mathbb{N}$, of finite rank, with φ and Γ, such that:

- Γ acts trivially on $\mathbb{N} / X.\mathbb{N}$
- $(\det \mathrm{Mat}\, \varphi) = (Q^h)$ for some $h \geq 0$
- the (φ,Γ)-module $\mathcal{E}^t \otimes \mathbb{N}$ is étale.

If $\mathbb{N}$ is a Wach module, we define:

$$\mathrm{Fil}^j \mathbb{N} = \{ y \in \mathbb{N},\ \varphi y \in Q^j \mathbb{N} \}$$

The $\mathbb{Q}_p$-vector space $\mathbb{N}/X\mathbb{N}$ then becomes a filtered φ-module, with the induced Fil and φ.

Theorem: If $\mathbb{N}$ is a Wach module, then the p-adic rep$^{\underline{n}}$ V associated to $\mathcal{E}^t \otimes \mathbb{N}$ is crystalline and $D_{\mathrm{cris}}(V) \simeq \mathbb{N}/X\mathbb{N}$

One gets all crystalline reps (such that $\mathrm{Fil}^0 D_{\mathrm{cris}}(V) = D_{\mathrm{cris}}(V)$) in this way.

The resulting functor: $\{\text{Wach modules}\} \to \{\text{effective crystalline reps}\}$ is then an equivalence of categories.

If V is crystalline, then $\mathbb{D}(V) = \mathcal{R} \otimes \mathbb{N}(V)$ and $\mathbb{D}(V) = \mathcal{O}b(D_{cris}(V))$

We have:

$$\mathcal{R}^+ \otimes \mathbb{N}(V) = \{ y \in \mathcal{R}^+[\tfrac{X}{t}] \otimes D_{cris}(V),$$
$$\imath_n(y) \in \mathrm{Fil}^0(K_n[[t]] \otimes D_{cris}(V)) \ \forall n \geq 1 \}$$

In order to cut out $\mathbb{N}(V)$ from $\mathcal{R}^+ \otimes \mathbb{N}(V)$, we need to look at the "order of growth" of certain elements.

Choose some $0 < \rho < 1$; we say $f(x) \in \mathcal{R}^+$ is of order $\leq s$ if

$$\{ p^{-ns} \| f(x) \|_{0 \leq |x|_p \leq \rho^{1/p^n}} \}_{n \geq 0}$$

is bounded (this does not depend on ρ). Suppose that on $D_{cris}(V)$ the matrix of φ is $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix}$ in some basis $e_1, \dots, e_d$. We say that $y = \sum_{i=1}^{d} y_i \otimes e_i$ is of order $\leq s$ if y_i is of order $\leq s - v_p(\lambda_i)$.

Finally, note that $\mathrm{order}(y) \leq s \Leftrightarrow \mathrm{order}(y \cdot t) \leq s+1$ so we extend the notion to $\mathcal{R}^+[\frac{1}{t}]$.

theorem | We have $\mathbb{N}(V) = \{ y \in \mathcal{R}^+ \otimes \mathbb{N}(V),\ y \text{ is of order } \leq 0 \}$.

This allows us to describe elements of $\mathbb{N}(V)$ as functions with values in $D_{cris}(V)$, with conditions on their ~~zeroes~~ (or poles) at the $\zeta_{p^n} - 1$, and some growth conditions.

9 The operator ψ

Recall that $\varphi : \mathcal{R} \to \mathcal{R}$ is injective but not surjective: $\mathcal{R}$ is a free $\varphi(\mathcal{R})$-module of rank p.

$$\mathcal{R} = \varphi(\mathcal{R}) \oplus (1+X)\varphi(\mathcal{R}) \oplus \cdots \oplus (1+X)^{p-1}\varphi(\mathcal{R})$$

so that if $y \in \mathcal{R}$, we can write:

$$y = \varphi(y_0) + (1+X)\varphi(y_1) + \cdots + (1+X)^{p-1}\varphi(y_{p-1})$$

and we set $\psi(y) = y_0$.

If $\mathbb{D}$ is a (φ,Γ)-module $/\mathcal{R}$ then $\mathbb{D} = \bigoplus_{i=0}^{p-1} (1+X)^i \varphi(\mathbb{D})$ and if $y = \sum_{i=0}^{p-1} (1+X)^i \varphi(y_i)$, we set $\psi(y) = y_0$. We have:

- $\psi \circ \varphi = \mathrm{id}$
- in particular, ψ is surjective
- $\psi \circ \gamma = \gamma \circ \psi$

Theorem: If $\gamma \in \Gamma$, then $1-\gamma : \mathbb{D}^t(V)^{\psi=0} \longrightarrow \mathbb{D}^t(V)^{\psi=0}$ is invertible.

corollary: The complexes $C^\bullet_\varphi(\mathbb{D}^t(V))$ and $C^\bullet_\psi(\mathbb{D}^t(V))$ are quasi-isomorphic.

Using this we get a map $h^1_{K_n} : \mathbb{D}^t(V)^{\psi=1} \longrightarrow H^1(K_n, V)$ $(K_n = \mathbb{Q}_p(\zeta_{p^n}))$

The Iwasawa cohomology of V is $H^1_{Iw}(\mathbb{Q}_p, V) = \mathbb{Q}_p \otimes H^1_{Iw}(\mathbb{Q}_p, T)$,

$$H^1_{Iw}(\mathbb{Q}_p, T) = \varprojlim H^1(K_n, T)$$

Using the above maps, we can prove:
$$\begin{aligned} \mathbb{D}^t(V)^{\psi=1} &\simeq H^1_{Iw}(\mathbb{Q}_p, V) \\ y &\mapsto (h^1_{K_n}(y))_{n\geq 0} \end{aligned}$$

Now let $\mathbb{D}$ be a (φ,Γ)-module over $\mathcal{O}_{\mathcal{E}}$ (eg: $\mathbb{D} = \mathbb{D}(T)$ where T is a $\mathbb{Z}_p$-rep), and let M be a $\mathbb{Z}_p[[X]]$-submodule of $\mathbb{D}$. We say that M is a Σ-lattice in $\mathbb{D}$ (a "treillis") if M is compact and if the image of M in $\mathbb{D}/p\mathbb{D}$ is a $\mathbb{F}_p[[X]]$-lattice.

Theorem: If $\mathbb{D}$ is a (φ,Γ)-module over $\mathcal{O}_{\mathcal{E}}$, then $\exists!$ a Σ-lattice $\mathbb{D}^\# \subset \mathbb{D}$ such that:

- $\forall y \in \mathbb{D}$ and $k \geq 0$, $\exists\, n_0$ st $\psi^n(y) \in \mathbb{D}^\# + p^k\mathbb{D}$ if $n \geq n_0$
- $\psi : \mathbb{D}^\# \longrightarrow \mathbb{D}^\#$ is onto

If $\mathbb{D} = \mathbb{D}(T)$ then $\mathbb{D}^\# \subset \mathbb{D}^t(T)$.

example: if V is a crystalline rep such that $\mathrm{Fil}^0 D_{cris}(V) = D_{cris}(V)$ and $\mathrm{Fil}^{h+1} D_{cris}(V) = 0$ then $\mathbb{N}(V) \subset \mathbb{D}^\#(V) \subset X^{-h-1}\,\mathbb{N}(V)$, where $\mathbb{D}^\#(V) = \mathbb{D}^\#(T)[\frac{1}{p}]$.

Now let T be a $\mathbb{Z}_p$-rep$^{\underline{n}}$ of $G_{\mathbb{Q}_p}$, $\mathbb{D}(T)$ be the associated (φ,Γ)-module over $\mathcal{O}_{\mathcal{E}}$, and $\mathbb{D}^\#(T)$ be Colmez' Σ-lattice.

Define: $\psi^{-\infty}(\mathbb{D}^\#(T)) = \{ (v_n)_{n\geq 0}\ ,\ v_n \in \mathbb{D}^\#(T)$, such that:

- $\forall k \geq 1$, $\exists$ a $\mathbb{Z}_p/p^k[[X]]$-lattice Λ_k of $\mathbb{D}(T)/p^k$ such that $v_n \in \Lambda_k\ \forall n$ ("$(v_n)_{n\geq 0}$ bounded")
- $\psi(v_{n+1}) = v_n$ $\}$

let $M(\mathbb{Q}_p) = \{ \left(\begin{smallmatrix} 1 & z \\ 0 & a \end{smallmatrix}\right) \in GL_2(\mathbb{Q}_p) \}$ = the mirabolic subgroup

If $g \in M(\mathbb{Q}_p)$, we can write:

$$g = \begin{pmatrix} 1 & \\ & p^j \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ & a \end{pmatrix} \cdot \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix}$$
$$j \in \mathbb{Z} \qquad a \in \mathbb{Z}_p^\times \qquad z \in \mathbb{Q}_p$$

and if $v = (v_n)_{n\geq 0} \in \psi^{-\infty}(\mathbb{D}^\#(T))$, we set:

$$\left[\begin{pmatrix} 1 & \\ & p^j \end{pmatrix} \cdot v \right]_i = v_{i-j} = \psi^j(v_i)$$
$$\left[\begin{pmatrix} 1 & \\ & a \end{pmatrix} \cdot v \right]_i = \gamma_a^{-1}(v_i) \quad \text{where } \chi(\gamma_a) = a$$
$$\left[\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} \cdot v \right]_i = (1+X)^z v_i$$
$$= \psi^j\left((1+X)^{p^j z} v_{i+j} \right) \quad \text{for } j \gg 0.$$

This gives a continuous action of $M(\mathbb{Q}_p)$ on $\psi^{-\infty}(\mathbb{D}^\#(T))$. We get a functor:

$$T \longmapsto \psi^{-\infty}(\mathbb{D}^\#(T))$$
$$\{\mathbb{Z}_p\text{-reps}\} \longrightarrow \{\text{reps of } M(\mathbb{Q}_p)\}$$

which is fully faithful: Colmez has shown how to recover T from $\psi^{-\infty}(\mathbb{D}^\#(T))$.

~~The second half of the course will be devoted to a discuss~~

Theorem (Colmez' construction of the p-adic Langlands correspondence)

If V is a trianguline rep$^{\underline{n}}$ of dim 2 of $G_{\mathbb{Q}_p}$ and $B(V) = \left[\mathbb{Q}_p \otimes \psi^{-\infty}(\mathbb{D}^\#(T)) \right]^{\text{dual}}$ then the action of $M(\mathbb{Q}_p)$ on $B(V)$ extends to an action of $GL_2(\mathbb{Q}_p)$, which gives the representation predicted by Breuil if V is de Rham.

The action of $M(\mathbb{Q}_p)$ on $B(V)$ is topologically irreducible if V is irreducible.

The second half of the course will be devoted to a discussion of this construction.

⑩ The mod p correspondence (some examples)

To illustrate the constructions of the preceding chapter, I'll describe $(\psi^{-\infty}(\mathbb{D}^\#(W)))^{dual}$ for W a $\mathbb{F}_p$-repⁿ of dim 1.

In this chapter, we can use coefficients (k_L = finite / $\mathbb{F}_p$).

All characters $G_{\mathbb{Q}_p} \to k_L^\times$ are of the form $\omega^r \mu_y$

where ω = mod p cyclo. character

μ_y = unramified character : $\mathrm{Frob}_p \mapsto y \in k_L^\times$

let W = associated 1-dimˡ repⁿ

$\mathbb{D}(W) = k_L((X)) \cdot e$ where $\varphi(e) = y \cdot e$
$\gamma(e) = \omega^r(\gamma) \cdot e$

$\mathbb{D}^+(W) = k_L[[X]] \cdot e$ is stable under φ and Γ

and $\mathbb{D}^\#(W) = X^{-1}\mathbb{D}^+(W)$

indeed, if $j \geq 1$ then $\psi(X^{-(j+1)}\mathbb{D}^+(W)) \subset X^{-j}\mathbb{D}^+(W)$
and $\psi(X^j \mathbb{D}^+(W)) \supset X^{j-1}\mathbb{D}^+(W)$

Note that $\psi^{-\infty}(\mathbb{D}^+(W)) \subset \psi^{-\infty}(\mathbb{D}^\#(W))$ is stable under $M(\mathbb{Q}_p)$

Let χ be a smooth character of $\mathbb{Q}_p^\times$ and define an action of the Borel subgroup $B(\mathbb{Q}_p)$ on $\psi^{-\infty}(\mathbb{D}^\#(W))$ by:

$$\left[\begin{pmatrix} a & \\ & a \end{pmatrix} v\right]_i = \chi^{-1}(a) \cdot v_i$$

in addition to the action of $M(\mathbb{Q}_p)$. let $\Omega_\chi(W) = [\psi^{-\infty}(\mathbb{D}^+(W))]^{dual}$

This is a smooth irreducible repⁿ of $B(\mathbb{Q}_p)$.

If χ_1, χ_2 are smooth characters of $\mathbb{Q}_p^\times$, define:

$$\chi_1 \otimes \chi_2 : B(\mathbb{Q}_p) \longrightarrow k_L^\times$$
$$\begin{pmatrix} a & b \\ & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d)$$

let η_W = smooth character of $\mathbb{Q}_p^\times$ associated to W by class field theory. Finally, let res : $\psi^{-\infty}(\mathbb{D}^\#(W)) \to k_L$ be the map $(y_0, y_1, \dots) \mapsto$ coeffᵗ of $\frac{1}{X}$ in y_0

theorem: The map res induces an exact sequence:

$$0 \to \chi \cdot \omega \cdot \eta_W^{-1} \otimes \omega^{-1} \eta_W \to [\psi\text{-}\infty(\mathbb{D}^{\#}(W))]^{\text{dual}} \to \mathcal{B}_{\chi}(W) \to 0$$

Finally, define:

$$\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2) = \{ \sigma : GL_2(\mathbb{Q}_p) \to k_L \ , \ \sigma \text{ locally constant}, \ \sigma(bg) = (\chi_1 \otimes \chi_2)(b) \cdot \sigma(g) \}$$

and let $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$ be the subrepresentation of σ's such that $\sigma(\mathrm{Id}) = 0$.

theorem: We have $\mathcal{B}_{\chi}(W) = \mathrm{Ind}_B^G(\eta_W \otimes \chi \eta_W^{-1})_0$