TALK 2

Serre-type Conj., mod p coho., and rep. theory of $GL_n(\mathbb{Q}_p)$

(Florian Herzig)

① $\boxed{\text{Compact unitary gps.}}$

$E/\mathbb{Q}$ imag. quad., split at p, $\quad \mathrm{Gal}(E/\mathbb{Q}) = \langle c \rangle$

$\langle , \rangle$: pos. def. hermitian form on E^n.

$G/\mathbb{Q}$: unitary gp. of $\langle , \rangle$:

$$\forall\ \mathbb{Q}\text{-alg. } A, \quad G(A) = \{ g \in GL_n(A \otimes_{\mathbb{Q}} E) : g \text{ preserves } \langle , \rangle \}.$$

Then $G \times_{\mathbb{Q}} \mathbb{R} \cong U(n)$, and $\underbrace{G \times_{\mathbb{Q}} \mathbb{Q}_\ell \cong GL_n}_{\substack{\text{fix. isos.} \\ \leftrightarrow \text{fix } \lambda | \ell.}}$ $\forall \ell$ split in E.
(in particular, $\ell = p$)

Let $U \subset G(\mathbb{A})$ be cpt. $\overset{\text{open}}{\text{subgp.}}$

$$\begin{aligned} U &= U_\infty \times U_p \times U^{\infty,p} \quad \text{with } U_\infty = G(\mathbb{R}) \quad \text{(cpt!)} \\ & U_p = GL_n(\mathbb{Z}_p) \quad \text{("level prime to } p\text{")} \\ & U^{\infty,p} \text{ suff. small} \end{aligned}$$

Have cov. space:

$$\begin{tikzcd} X_U^p := G(\mathbb{Q}) \backslash G(\mathbb{A}) / U^p \circlearrowleft G(\mathbb{Q}_p) = GL_n(\mathbb{Q}_p). \arrow[d] \\ X_U := G(\mathbb{Q}) \backslash G(\mathbb{A}) / U \end{tikzcd}$$

(0-dim^l loc. symm. space)

For W a rep. of U_p, have loc. const. sheaf $\mathcal{L}_W$ on X_U.

$$M(U, W) := H^0(X_U, \mathcal{L}_W) = \mathrm{Hom}^{\text{cont}}_{U_p}(X_U^p, W) \quad \underline{\text{f.d.}}$$

$$\circlearrowleft$$

"alg. modular forms of wt. W" (Gross)

$\mathbb{T} := \mathbb{Z}[T_{\ell,1}, \ldots, T_{\ell,n} :$
ℓ split in E, $\ell \neq p$,
$U_\ell = GL_n(\mathbb{Z}_\ell)]$

Hecke: Write $GL_n(\mathbb{Z}_\ell)\begin{pmatrix} \overbrace{\ell \ \ddots \ \ell}^{i} & & \\ & 1 & \\ & & \ddots \ 1 \end{pmatrix} GL_n(\mathbb{Z}_\ell) = \coprod_{\text{fin.}} \gamma_\alpha GL_n(\mathbb{Z}_\ell)$.

Then $(T_{\ell,i} f)(x) = \sum_\alpha f(x\gamma_\alpha)$. (here $f : X_U^p \to W$)

Two choices for W:

① W a rep. of $G \times_{\mathbb{Q}} \mathbb{Q}_p \cong GL_n$:

a $\mathbb{T}$-evec. $f \in M(U, W)$ has an attached

Galois rep. $\rho_f : G_E \to GL_n(\overline{\mathbb{Q}}_p)$ s.t. $\rho_f \circ c \cong \rho_f^\vee$

→ known for $n=3$: Rogawski, etc.
and general n (under certain local conditions)
Kottwitz, Clozel

② $W = F(\lambda)$ a Serre weight:

$$U_p = GL_n(\mathbb{Z}_p) \twoheadrightarrow GL_n(\mathbb{F}_p) \curvearrowright F(\lambda).$$

Using $M(U, F(\lambda)) \subset M(U, W(\lambda))$ and lifting to char. 0

(D-S): $\mathbb{T}$-evec. $f \in M(U, F(\lambda)) \rightsquigarrow \rho_f : G_E \to GL_n(\overline{\mathbb{F}}_p)$,
$\rho_f \circ c \cong \rho_f^\vee$.

→ known: see above remarks

Rk: Frob_ℓ for good ℓ are dense in G_E!

- Serre-type Conj.:

$\rho : G_E \to GL_n(\overline{\mathbb{F}}_p)$ irr., $\rho \circ c \cong \rho^\vee$

$\rightsquigarrow W(\rho) = \{$ Serre wts. $F \mid \rho$ attached to $\mathbb{T}$-evec. in
$M(U, F)$, some U as above $\}$

Same $W^?(\rho)$ as in prev. talk.

Conj.: $W(\rho) = W^?(\rho)$ if ρ occurs in cohomology
i.e. $H^0(X_U^p, \overline{\mathbb{F}}_p)[m_\rho] \neq 0$ for some U^p
[there could be local obstructions for $\ell \neq p$]

Different viewpoint:

$$\mathcal{M}(U,F) = H^0(X_U, \mathcal{L}_F) \cong \mathrm{Hom}_{U_p}(F^\vee, \underset{\substack{\circlearrowleft \\ G(\mathbb{Q}_p)}}{H^0(X^p_U, \overline{\mathbb{F}}_p)}) \qquad (*)$$

<u>Note:</u> $$H^0(X^p_U, \overline{\mathbb{F}}_p) = \varinjlim_{U_p' \subset U_p \text{ open}} H^0(X_{U^pU_p'}, \overline{\mathbb{F}}_p)$$

as in Emerton's talk.

Let $m_\rho \trianglelefteq \mathbb{T}$ be max. ideal assoc. to ρ.

$$W(\rho) = \{ F : F^\vee \underset{U_p\text{-lin.}}{\hookrightarrow} H^0(X^p_U, \overline{\mathbb{F}}_p)[m_\rho], \text{ some } U \text{ as above} \}.$$

② Hecke action at p

From $(*)$,

$$\mathcal{M}(U,F) \cong \mathrm{Hom}_{G(\mathbb{Q}_p)}(\underbrace{\overline{\mathbb{F}}_p[G(\mathbb{Q}_p)] \otimes_{\overline{\mathbb{F}}_p[U_p]} F^\vee}, H^0(X^p_U, \overline{\mathbb{F}}_p))$$

U_p cpt. open

$$\cong \{ \text{functions } \varphi : G(\mathbb{Q}_p) \to F^\vee$$
$$\cdot\ \varphi(ug) = u\varphi(g) \quad \forall u \in U_p,\ g \in G(\mathbb{Q}_p)$$
$$\cdot\ \varphi \text{ loc. const., cpt. supp.} \}.$$

$$=: \mathrm{ind}_{U_p}^{G(\mathbb{Q}_p)}(F^\vee) \qquad \text{(\underline{compact induction})}$$

$\therefore$ The $\overline{\mathbb{F}}_p$-algebra $\mathcal{H}_{U_p}(F^\vee) := \mathrm{End}_{G(\mathbb{Q}_p)}(\mathrm{ind}_{U_p}^{G(\mathbb{Q}_p)} F^\vee)$

acts on $\mathcal{M}(U,F)$, commuting with $\mathbb{T}$-action.

Notation: $K := U_p$, $G := G(\mathbb{Q}_p)$.

Viewpoints:

(i) Biinvariant functions:

$$\mathcal{H}_K(F) = \mathrm{Hom}_G(\mathrm{ind}_K^G F, \mathrm{ind}_K^G F) = \mathrm{Hom}_K(F, \underbrace{\mathrm{ind}_K^G F}_{\text{maps } G \to F})$$

$$\cong \left\{ f : G \to \mathrm{End}\, F : \begin{array}{ll} \cdot\ f(k_1 g k_2) = k_1 f(g) k_2 & \forall k_i \in K \\ & g \in G \\ \cdot\ \text{loc. const., cpt. supp.} & \end{array} \right.$$

under convolution:

$$(f_1 * f_2)(g) = \sum_{\gamma \in G/K} f_1(g\gamma) f_2(\gamma^{-1})$$

Note: $\mathcal{H}_K(\mathbb{1}) \cong \overline{\mathbb{F}}_p[K \backslash G / K]$

(ii) Yoneda:

$\mathcal{H}_K(F)^{op}$ consists of endos. of functor

$$\begin{array}{c} \mathrm{Hom}_G(\mathrm{ind}_K^G F, -) : \underline{G\text{-mod}} \longrightarrow \underline{\text{set}} \\ \shortparallel \\ \mathrm{Hom}_K(F, -) \end{array} \qquad [F = \mathbb{1} : \pi \longmapsto \pi^K]$$

Structure:

Let $\bar{N} := \begin{pmatrix} 1 & & \\ & \ddots & \\ * & & 1 \end{pmatrix}$

Basic fact: $F^{\bar{N}(\mathbb{F}_p)}$ is one dim^l. (lowest wt. space)

$\circlearrowleft$

$T(\mathbb{F}_p)$ acts by char. χ_F.

Let $T^+ := \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \in T : \operatorname{ord}(t_1) \geq \cdots \geq \operatorname{ord}(t_n) \right\}$.

Thm. (H.)

There is an inj. homo. of $\overline{\mathbb{F}}_p$-algebras

$$\begin{tikzcd} \mathcal{H}_K(F) \arrow[r, hook, "\text{“Satake”}"] & \mathcal{H}_{T(\mathbb{Z}_p)}(\chi_F) \\ f \arrow[r, mapsto] & \left(t \mapsto \left(\sum_{\bar{n} \in \bar{N}/\bar{N}(\mathbb{Z}_p)} f(t\bar{n}) \right) \Big|_{F^{\bar{N}(\mathbb{F}_p)}} \right) \end{tikzcd}$$

with image

$\mathcal{H}^+_{T(\mathbb{Z}_p)}(\chi_F) := \{ \varphi : T \longrightarrow \overline{\mathbb{F}}_p :$

- $\varphi(t_0 t) = \chi_F(t_0)\varphi(t) \quad \forall t_0 \in T(\mathbb{Z}_p), t \in$
- loc. const., cpt. supp.
- $\operatorname{supp} \varphi \subset T^+ \}$.

$\cong \overline{\mathbb{F}}_p[Y(T)_+]$

$\uparrow$ choice of uniformiser

Cor.: $\mathcal{H}_K(F)$ comm.

Rk: Same formula as in classical case ($/\mathbb{C}$), but drop modulus char. (power of p).

Classically, image $\cong \mathbb{C}[Y(T)]^W$. (here toric)

Proof outline:

Cartan dec.: $G = \coprod_{\mu \in Y(T)_+} K\mu(p)K$

Let $\mu \in Y(T)_+$, $t = \mu(p)$.

red : $K = GL_n(\mathbb{Z}_p) \twoheadrightarrow GL_n(\mathbb{F}_p)$

Lemma: red$(K \cap K^t) = P(\mathbb{F}_p)$, where $P = MU$ is parab. associated to μ.

red$({}^tK \cap K) = \bar{P}(\mathbb{F}_p)$, opp. parab.

The map $F^{\bar{U}(\mathbb{F}_p)} \hookrightarrow F \twoheadrightarrow F_{U(\mathbb{F}_p)}$ is an iso. of irr. $M(\mathbb{F}_p)$-reps.

Find: $\exists!\, T_\mu \in \mathcal{H}_K(F)$ s.t.

(a) supp $T_\mu = K\mu(p)K$

(b) $T_\mu(\mu(p)) : F \twoheadrightarrow F_{U(\mathbb{F}_p)} \xleftarrow{\sim} F^{\bar{U}(\mathbb{F}_p)} \hookrightarrow F$

Also: (a) determines T_μ up to scalar.

Then proceed as in classical case. □

Prop. (H.)
Suppose $F = F(\lambda)$, $\lambda \in X_1(T)$ with $\langle \lambda, \alpha_i^\vee \rangle \neq 0$ $\forall i$.
Then $T_\mu T_\nu = T_{\mu+\nu}$ $\forall \mu, \nu$.

$G = GL_n$: $\mu_i(t) = \begin{pmatrix} t & & & & \\ & \ddots & & & \\ & & t & & \\ & & & 1 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$ with $\underbrace{t,\dots,t}_{i}$ in $Y(T)_+$ ("basis")

$T_i := T_{\mu_i}$. Then $\mathcal{H}_K(F) \cong \bar{\mathbb{F}}_p[T_1, \dots, T_{n-1}, T_n^{\pm 1}]$.

$n = 2$: Barthel-Livné

Comparison with classical Hecke operators

$W(\lambda)_{\overline{\mathbb{Z}}} \subset W(\lambda)_{\overline{\mathbb{Q}}}$ - - $U_p = GL_n(\mathbb{Z}_p)$ - stable lattice.

$\circlearrowleft$ with reduction $\mathcal{M}(U, W(\lambda)_{\overline{\mathbb{F}}_p})$.

T_μ^{cl}

$$(T_\mu^{cl} f)(x) = \sum_{g \in K\mu(p)K/K} g\, f(xg)$$

③ Application to Serre-type Conjectures

$\rho : G_E \longrightarrow GL_n(\overline{\mathbb{F}}_p)$ irr., $\rho \circ c \cong \rho^\vee$

suppose $F = F(\overbrace{a_1, \dots, a_n}^{\lambda}) \in W(\rho)$

$$\therefore \quad \mathcal{M}(U, F)[\mathfrak{m}_\rho] \neq 0 \qquad (\text{some } U)$$
$$\cup$$
$$\mathcal{H}_K(F^\vee) \qquad\qquad [\text{f.d.}]$$

Prop. (EGH) Suppose $n = 3$ (to be safe).
If $\exists\ 0 \neq f \in \mathcal{M}(U, F)[\mathfrak{m}_\rho]$ s.t. $T_i f = \xi_i f$, $\xi_i \neq 0$ $(1 \leq i \leq n)$

then
$$\rho|_{I_p} \sim \begin{pmatrix} \omega^{a_1+n-1} & & & * \\ & \ddots & & \\ & & \omega^{a_{n-1}+1} & \\ & & & \omega^{a_n} \end{pmatrix}.$$

Proof: By prev. prop. and Deligne-Serre lifting lemma, there is an evec. $\tilde{f} \in \mathcal{M}(U, W(a_1, \dots, a_n)_{\overline{\mathbb{Q}}_p})$ for $\mathbb{T}$ and the $p^{-\langle w_0\lambda, \mu\rangle} T_\mu^{cl}$ $(\mu \in Y(T)_+)$ s.t. the evals. lift those of $\mathbb{T}$ and T_μ on f.

$\therefore$ T_i^{cl}-eval. on $\tilde{f}$ has valuation $\langle w_0\lambda, \mu_i\rangle = a_n + \dots + a_{n+1-}$

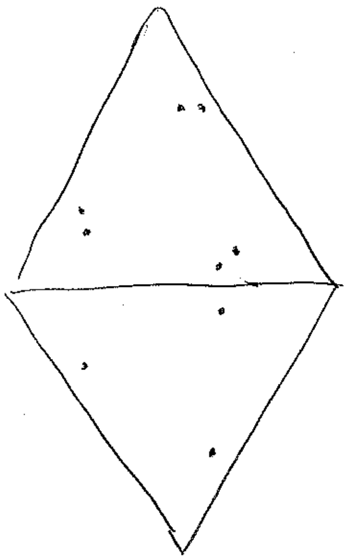
The associated Galois rep. $\rho_{\tilde{f}} : G_E \longrightarrow GL_n(\overline{\mathbb{Q}}_p)$ lifts ρ, is crys. at p, has HT wts. $(a_1+n-1, \dots, a_n)$ and $\varphi \curvearrowright D_{cris}(\rho_{\tilde{f}}|_{D_p})$ has slopes $a_1+n-1, \dots, a_n$.

Thus $\rho_{\tilde{f}}|_{D_p}$ is ordinary and so $\rho_{\tilde{f}}|_{I_p} \sim \begin{pmatrix} \chi^{a_1+n-1} & & * \\ & \ddots & \\ & & \chi^{a_n} \end{pmatrix}$.

(χ = p-adic cyclo.) □

Rk: Similar, but weaker, result if only some of the T_i have non-zero eval.

E.g.: $n = 3$, tame $\rho|_{I_p}$ of niveau 1. + generic



The 6 obv. wts. should be ordinary [can guess now what they are...]

The 3 shadows have to be supersing. ($T_1 = T_2 = 0$)