TALK 1

Serre-type Conj. for tame n-dimensional mod p Gal. reps.

(Florian Herzig)

- Notations: p prime

Fix $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$

$G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$\cup$

D_p = decomp. gp.

$\cup$

I_p = inertia gp.

① $\underline{GL_2}$ $\quad k \geq 2$

$f \in S_k(\Gamma_1(N), \mathbb{C})$ eigenform : $T_\ell f = a_\ell f$, $S_\ell f = b_\ell f$

$\circlearrowleft T_\ell$, $\circlearrowleft S_\ell$ $\quad (\ell \nmid pN)$ $\qquad a_\ell, b_\ell \in \overline{\mathbb{Z}}$

$\rightsquigarrow \rho_f : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ unram. outside Np s.t.

(i) $\det(1 - \rho_f(\mathrm{Frob}_\ell) X) = 1 - \bar{a}_\ell X + \ell \bar{b}_\ell X^2 \quad \forall \ell \nmid pN$

(ii) $\rho_f(\text{cx. conj.}) \sim \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ if $p > 2$ $\quad$ ("$\underline{\text{odd}}$").

$\underline{\text{Serre's Conjecture:}}$ If $\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ is irr. + odd, it arises from some f of level $N(\rho)$ and weight $k(\rho) \geq 2$

$N(\rho) \leftarrow \{\rho|_{I_\ell} : \ell \neq p\}$ (prime to p)

$k(\rho) \leftarrow \{\rho|_{I_p}\}$.

Passing to cohomology:

Eichler-Shimura: $H^1(\Gamma_1(N), \mathrm{Sym}^{k-2}\mathbb{C}^2) \cong M_k(N) \oplus \overline{S_k(N)}$

(group coho., via $\Gamma_1(N) \subset SL_2(\mathbb{Z}) \curvearrowright \mathbb{C}^2$) (Hecke equiv.)

For $\rho : G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{F}}_p)$ irr.,

ρ attached to reduction "mod p" of Hecke evals. in wt. k, level prime to p

$\Leftrightarrow$ - - - - - - - - Hecke evals. in $H^1(\Gamma_1(N), \mathrm{Sym}^{k-2}\overline{\mathbb{F}}_p^2)$, some $(N,p)=1$.

(*) $\{ \Leftrightarrow$ - - - - - - - - - - - - - - - - $H^1(\Gamma_1(N), F)$ - - - - - - - some $F \in JH(\mathrm{Sym}^{k-2}$

$\rightarrow$ Ash–Stevens 1986

Serre weights = irr. reps. of $GL_2(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$

$F(a,b) \cong \mathrm{Sym}^{a-b}\overline{\mathbb{F}}_p^2 \otimes \det^b \qquad (0 \le a-b \le p-1)$

(So there are $p(p-1)$.)

Let $W(\rho) = \{ \text{Serre wts. } F \mid (*) \text{ holds} \}$.

Translate between $k(\rho)$ and $W(\rho)$ (exercise)

"$\rightarrow$": use
- $W(\rho\otimes\omega)\cong W(\rho)\otimes\det$
- ρ modular of wt. 2 $\Rightarrow$ wt. $p+1$
- ρ modular of wt. k $\Rightarrow$ k determined mod $p-1$ by $\det\rho|_{I_p}$

"$\leftarrow$": need to work out $JH(\mathrm{Sym}^{k-2}\,\mathbb{F}_p^2)$ for $k\le p^2-1$.

($\underline{\text{Rk}}$: $k(\rho)$ minimal wt. in prime-to-p level.)

$\underline{\text{E.g.}}$: $\rho|_{I_p}\sim\begin{pmatrix}\omega^i & * \\ & \omega^j\end{pmatrix}$: $p-2>i>j+1>0$

$k(\rho)=i+pj+1$

$W(\rho)=\begin{cases}\{F(i-1,j),\,F(j+p-2,i)\} & *\ \text{split}\\ \{F(i-1,j)\} & *\ \text{non-split}\end{cases}$

(see also BDJ, thm. 3.15)

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② $\underline{GL_n}$, $n\geq 2$

$\Gamma_1(N)=\left\{\gamma\in SL_n(\mathbb{Z}):\ \gamma\equiv\begin{pmatrix} & * & \\ 0 & \cdots\ 0 & 1\end{pmatrix}\bmod N\right\}$, $(N,p)=1$

F ... Serre weight: irr. rep. of $GL_n(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$.

$\rightsquigarrow\ H^i(\Gamma_1(N),F)$

$\circlearrowleft$

$T_{\ell,1},\dots,T_{\ell,n}$ $(\forall\,\ell\nmid pN)$.

If $\rho: G_{\mathbb{Q}}\to GL_n(\overline{\mathbb{F}}_p)$ $\underline{\text{irr.}}$ + $\underline{\text{odd}}$ (i.e. $\rho(\text{cx. conj.})\sim\begin{pmatrix}\pm1 & & \\ & \mp1 & \\ & & \pm1 \\ & & & \ddots\end{pmatrix}$ if $p>2$)

let $W(\rho)=\{$Serre wts. $F\mid\exists$ Hecke eigenvec. in $H^i(\Gamma_1(N),F)$ attached to ρ, $\underline{\text{some}}$ $i\geq 0$, $(N,p)=1\}$.

$\rightarrow$ Ash, Doud, Pollack, Sinnott

Rks.: – $W(\rho)$ should only depend on $\rho|_{I_p}$.

– expect $\emptyset \neq W(\rho|_{I_p}) \subset W(\rho|_{I_p}^{ss})$

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③ Modular representations

$G = GL_n$ [or split reductive $/\mathbb{F}_p$, G' simply conn.]

$\cup$

B Borel $\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$

$\cup$

T max. torus $\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$

$X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$, $Y(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ free $/\mathbb{Z}$ of rk. n

$\langle , \rangle : X(T) \times Y(T) \longrightarrow \mathbb{Z}$ perfect

$R^+ \subset R \subset X(T)$ (pos.) roots

$\alpha \in R \longleftrightarrow \alpha^\vee \in R^\vee \subset Y(T)$

simple coroots $\alpha_i^\vee(t) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & t & & \\ & & & t^{-1} & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$, $1 \le i \le n-1$

$\quad\uparrow i$

dominant wts. $X(T)_+$ $(a_1 \geq \dots \geq a_n$ in $\mathbb{Z})$

Weyl gp. $W = N(T)/T \cong S_n$

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restricted wts. $X_1(T) = \{\lambda \in X(T) ; 0 \le \langle \lambda, \alpha_i^\vee \rangle \le p-1 \ \forall i\} \subset X(T)$

$\cup$

regular wts. [non-standard terminology] $X_{reg}(T) =$ – – – – – – – $0 \le$ – – $< p-1$ – – –

$\cup$

$X^0(T) =$ – – – – $0 =$ – – –

Thm. (Chevalley)

$$\begin{array}{ccc} \{\text{irr. reps. of } G_{/\overline{\mathbb{F}}_p}\} & \longleftrightarrow & X(T)_+ \\ \Psi & & \Psi \\ F(\lambda) & \longleftrightarrow & \lambda \end{array}$$

Thm. (Steinberg + ε)

(i) Every irr. rep. of $G(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$ (Serre wt.) is of form $F(\lambda)$, some $\lambda \in X_1(T)$.

(ii) If $\lambda, \lambda' \in X_1(T)$, $F(\lambda) \cong F(\lambda')$ as reps. of $G(\mathbb{F}_p)$ $\iff$ $\lambda - \lambda' \in (p-1)X^0(T)$.

E.g. Serre wts. for $n = 3$: $F(a,b,c)$: $0 \leq a-b,\ b-c \leq p-1$.

reg. ———————— : ———— —————— $< p-1$.

$$F(\underline{a} + \underline{1}) \cong F(\underline{a}) \otimes \det.$$

Dual Weyl module: for $\lambda \in X(T)_+$, let

$$W(\lambda) = \{\text{morphisms } G(\overline{\mathbb{F}}_p) \xrightarrow[f]{} \overline{\mathbb{F}}_p : f(\bar{b}g) = \lambda(\bar{b}) f(g)$$
$$\forall \bar{b} \in \overline{B} \text{ (opp. Borel)},\ g \in G\}$$

(action by right translation)

→ works over any comm. ring

→ satisfies the Weyl character formula

→ $F(\lambda) \cong \mathrm{soc}_G W(\lambda)$ [unique irr. submod.]

E.g. ($n = 2$) $W(m, 0) \cong \mathrm{Sym}^m \overline{\mathbb{F}}_p^2$

Alcoves:

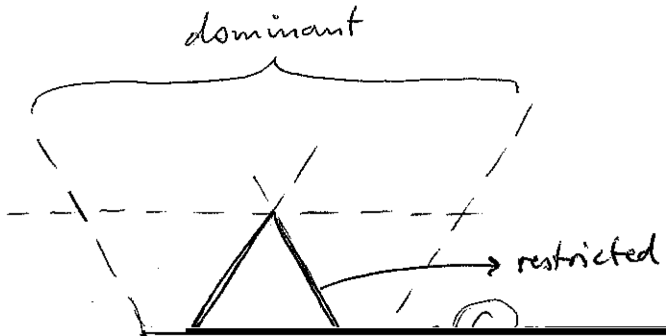
Choose $\tilde{\rho} \in X(T)$ s.t. $\langle \tilde{\rho}, \alpha_i^\vee \rangle = 1 \ \forall i$, e.g. $\tilde{\rho} = (n-1, n-2, \dots, 1, 0)$

For $\alpha \in R$, $m \in \mathbb{Z}$,

affine hyperplane $H_{\alpha,m} \subset X(T) \otimes \mathbb{R}$: $\langle \lambda + \tilde{\rho}, \alpha^\vee \rangle = mp$.

Alcoves = conn. components of $X(T) \otimes \mathbb{R} - \bigcup_{\alpha, m} H_{\alpha,m}$ (open!)

n = 3: [really the projection under $X(T) \otimes \mathbb{R} \twoheadrightarrow X(T \cap Sl_3) \otimes \mathbb{R}$]



$\boxed{\text{Assume for simplicity } C_0 \cap X(T) \neq \emptyset}$ $(p > n-1)$

<u>Affine Weyl group:</u> – generated by aff. reflections $s_{\alpha,m}$ in $H_{\alpha,m}$
– fund. domain C_0.

For $\lambda, \lambda' \in X(T)$, say $\lambda \uparrow \lambda'$ if $\exists$ aff. reflections $s_i = s_{\alpha_i, m_i}$ s.t.

$$\lambda \le s_1 \lambda \le s_2 s_1 \lambda \le \dots \le s_r s_{r-1} \cdots s_1 \lambda = \lambda'$$

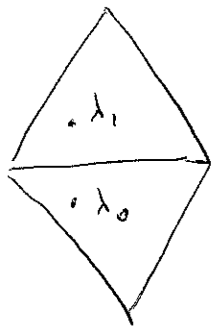
(in particular: λ, λ' in same aff. Weyl gp. orbit)

<u>Linkage Principle:</u> $[W(\lambda) : F(\mu)] \neq 0 \Rightarrow \mu \uparrow \lambda$
(multiplicity as JH constit.)

<u>Translation Principle:</u>
Suppose λ, μ lie in same aff. Weyl gp. orbit and on no $H_{\alpha,m}$. Then $[W(\lambda) : F(\mu)]$ only depends on the <u>alcoves</u> λ, μ lie in.

<u>Cor.:</u> $W(\lambda) = F(\lambda)$ if $\lambda \in C_0$. [consider highest wt.]

<u>Eg.</u> $(n = 3)$



If $a - c > p - 2$,
$a - b,\ b - c < p - 1$, then

$$0 \to \underset{(\lambda_1)}{F(a,b,c)} \to W(a,b,c) \to \underset{(\lambda_0)}{F(c+p-2, b, a-p+2)} \to 0$$

$(n = 2)$ If $p - 1 < m < 2p - 1$,

$$0 \to F(m, 0) \to W(m, 0) \to F(p-1, m-p+1) \to 0$$

Rk: It is possible that $\mu \uparrow \lambda$ with λ restricted and [redacted] restricted (e.g. GL_n, $n \geq 4$; G_2).

[redacted]

$\omega_r : I_p \longrightarrow \mathbb{F}_{p^r}^\times$ tame fundamental character

Assume for $\forall s \mid r$, $N_{\mathbb{F}_{p^r}/\mathbb{F}_{p^s}} \circ \omega_r = \omega_s$.

If $\mu = (a_1, \ldots, a_n) \in X(T)$, let $\bar{\mu} = (a_1, \ldots, a_n) \in Y(T)$.
[duality]

If $w \in W$, $\mu \in X(T)$: choose $r \geq 1$ s.t. $w^r = 1$.

let $\ldots \prod^{r-1} (\mathrm{Frob} \circ w)^i (\bar{\mu} \circ \omega_r) : I_p \to T(\mathbb{F}_{p^r}) \subset GL_n(\overline{\mathbb{F}}_p)$.

Ingredients for the conj:

(i) Given ρ tame at p:

$V(\rho|_{I_p})$ is a certain Deligne-Lusztig rep. of $GL_n(\mathbb{F}_p)$ over $\overline{\mathbb{Q}}_p$.

(genuine rep. up to sign, parabolic ind. of cuspidal)

(ii) $$\mathcal{R} : \{\text{Serre weights}\} \longrightarrow \underset{\substack{\wr\wr \\ (\mathbb{Z}/(p-1))^n}}{\{\text{regular Serre wts.}\}} \qquad \begin{array}{c} F(a_1, \ldots, a_n) \\ \downarrow \\ (\bar{a}_1, \ldots, \bar{a}_n). \end{array}$$

$$\mathcal{R}(F(b_1, \ldots, b_n)) := (\overline{b_n - (n-1)}, \ldots, \overline{b_2 - 1}, \bar{b}_1).$$

Conj. (H.) If $\rho : G_{\mathbb{Q}} \to GL_n(\overline{\mathbb{F}}_p)$ odd, irred. and $\rho|_{I_p}$ tame,
then
$$W_{reg}(\rho) = \underbrace{\mathcal{R}(JH(\overline{V(\rho|_{I_p})}))}_{=: W^?(\rho).}$$

regular Serre wts. in $W(\rho)$

→ prev. conjecture by Ash, Doud, Pollack, Sinnott for GL_3
(fewer wts. predicted, computations)

→ evidence:
- computations (Doud - Pollack)
- examples of ρ for $\underbrace{GL_3}_{\text{Ash et al}}$, GL_4 modular of certain predicted weight
- companion forms for GSp_4. (Tilouine)

Generic case: (use Jantzen)

Assume $\rho|_{I_p}$ tame and ($\rho|_{I_p}$ generic or ν generic).

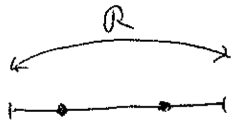
Then $F(\nu) \in W^?(\rho) \Leftrightarrow \left(\begin{array}{l} \exists\, \nu' \uparrow \nu \\ \text{dom. restricted} \\ \text{s.t. } \rho|_{I_p} \sim \tau(w, \nu' + \tilde{\rho}) \text{ for some } w \in W \end{array} \right.$

Terminology: $\nu' = \nu$: $F(\nu)$ is "obvious" weight (there are $\#W = n!$)

$\nu' \neq \nu$: $F(\nu)$ is "shadow" weight

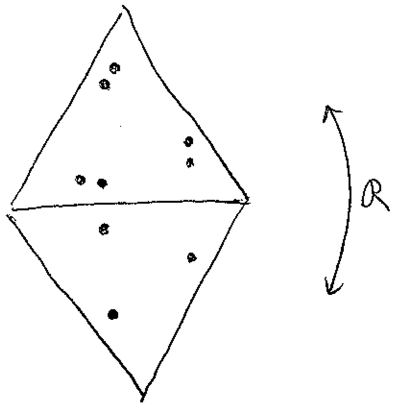
Examples:

GL_2:



2 obvious

GL_3:



6 obvious

3 shadow