Continuous representation theory of p-adic Lie groups

J. Teitelbaum and J.-F. Dat

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Today's lecture : First steps towards

- a Langlands correspondence for De Rham representations.
- a Langlands functoriality principle for "crystalline parameters".

Hypothetical p-adic Langlands correspondence for $L \supset \mathbb{Q}_p$:

$$\left\{ \begin{array}{c} \text{(certain) continuous } p\text{-adic} \\ \text{representations of } \mathit{Gal}(\overline{L}/L) \\ \text{of dimension } n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{(certain) continuous } p\text{-adic} \\ \text{representations of } \mathrm{GL}_n(L) \\ \text{(topologically irreducible)} \end{array} \right\}$$

Colmez' lecture : $n = 2$, $L = \mathbb{Q}_p$. In LHS all representations are considered, via Fontaine's theory of (φ, Γ)-modules. In RHS, one gets Banach representations.

Breuil's first insights : apply to De Rham representations only (and with distincts HT weights), via Fontaine's theory of filtered φ-modules. In RHS, also considers Banach representations.

Strategy for De Rham representations

There is a *mysterious* additive functor, due to Fontaine,

$$\left\{ \begin{array}{c} \text{All continuous } p\text{-adic} \\ \text{representations of } Gal(\overline{L}/L) \end{array} \right\} \xrightarrow{\mathrm{WD}} \left\{ \begin{array}{c} \text{“smooth” representations of} \\ \text{Weil-Deligne gp } \mathrm{WD}(\overline{L}/L) \end{array} \right\}$$

Fontaine : ρ in LHS is called De Rham if $dim(\mathrm{WD}(\rho)) = dim(\rho)$. To recover ρ from $\mathrm{WD}(\rho)$, what is “essentially” missing is an admissible filtration on (some base change of) $\mathrm{WD}(\rho)$.

First input : Classical Langlands correspondence associates some smooth irred. repr. $\pi_{\mathrm{WD}(\rho)}$ to $\mathrm{WD}(\rho)$.

Second input : Hodge-Tate weights (assumed distinct) “naturally” determine an irreducible algebraic representation $\pi_{\mathrm{HT}(\rho)}$ of GL_n.

Breuil's ideas :

- $\pi_{\mathrm{WD}(\rho)} \otimes \pi_{\mathrm{HT}(\rho)}$ should be contained in the locally algebraic vectors of the hypothetical π_ρ.
- Missing filtrations on $\mathrm{WD}(\rho)$ should “correspond” to Banach completions of the locally algebraic repr. $\pi_{\mathrm{WD}(\rho)} \otimes \pi_{\mathrm{HT}(\rho)}$.

What is (should be) crystalline functoriality ?

Let $\mathbf{G}$ be a reductive group over L and $\mathbf{G}'$ its Langlands dual. An hypothetical Langlands correspondence for $\mathbf{G}$ should look like

$$\left\{ \begin{array}{c} \text{continuous morphisms} \\ \mathit{Gal}(\overline{L}/L) \longrightarrow \mathbf{G}'(\overline{\mathbb{Q}}_p) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Continuous top. irr.} \\ \overline{\mathbb{Q}}_p - \text{representations of } \mathbf{G}(L) \end{array} \right\}$$

LHS is functorial in $\mathbf{G}'$, whence a "hidden" correspondence between RHS for different groups. This is *Langlands functoriality*.

In classical Langlands, functoriality is well understood for *unramified* representations, via so-called Satake isomorphism.

A De Rham representation is called crystalline if $\mathrm{WD}(\rho)$ is *unramified*.

Schneider-Teitelbaum's idea : crystalline functoriality should be related to suitable "completions" of Satake isomorphism.

Contents of this lecture

1. Banach Hecke algebras
2. p-adic Satake isomorphism
3. towards Crystalline correspondence
4. towards Crystalline functoriality and De Rham correspondence

Classical Hecke algebras (1/2)

Notation :

- G is a totally discontinuous locally compact group.
- $U \subset G$ is an open compact subgroup.
- $\rho : U \longrightarrow \mathrm{GL}(V)$ is a representation of U on a finite dimensional vec. sp. V over some field K.

To these data are associated the so-called Hecke algebra :

$$\mathcal{H}(G,\rho) := \left\{ \begin{array}{l} \text{compactly supported functions } \psi : G \longrightarrow \mathit{End}_K(V) \\ \forall u, u' \in U, \;\; \psi(ugu') = \rho(u) \circ \psi(g) \circ \rho(u') \end{array} \right\}.$$

whose product is given by

$$\psi_1 * \psi_2(g) = \sum_{x \in G/U} \psi_1(x) \circ \psi_2(x^{-1}g),$$

and whose unit is the function ψ_e supported on U defined by $\psi_e(u) = \rho(u)$.

Classical Hecke algebras (2/2)

This algebra is the endomorphism algebra of the compactly induced representation of ρ from U to G. More precisely, the compactly induced representation is

$$\mathit{ind}_U^G(\rho) := \left\{ \begin{array}{l} \text{compactly supported functions } f : G \longrightarrow V \\ \forall u \in U, \;\; f(gu) = \rho(u)^{-1}(f(g)) \end{array} \right\},$$

equipped with the left translation action of G. The following map

$$\begin{array}{rcl} \mathcal{H}(G,\rho) & \to & \mathit{End}_G(\mathit{ind}_U^G(\rho)) \\ \psi & \mapsto & \left(f \mapsto \psi * f : g \mapsto \sum_{x \in G/U} \psi(g^{-1}x)(f(x)) \right) \end{array}$$

is an isomorphism of K-algebras. The inverse isomorphism takes an endomorphism $A \in \mathit{End}_G(\mathit{ind}_U^G(\rho))$ to the function

$$\psi_A : g \mapsto \left(v \mapsto A(f_v)(g^{-1}) \right)$$

where $f_v \in \mathit{ind}_U^G(\rho)$ is the function which is supported on U and takes u to $\rho(u)^{-1}v$.

Completed Hecke algebras (1/2)

Further hypothesis and notation :

- The field K is a complete extension of $\mathbb{Q}_p$.
- The representation ρ is *continuous*.

Let $||.||$ be a U-invariant norm on the vec. sp. V, and denote also by $||.||$ the associated operator norm on $End_K(V)$.
Then the norm $||\psi|| := \sup_{g\in G} ||\psi(g)||$ on $\mathcal{H}(G,\rho)$ is submultiplicative, so the completion $\mathcal{B}(G,\rho)$ is a Banach algebra.
Explicitly we have :

$$\mathcal{B}(G,\rho) := \left\{ \begin{array}{l} \text{functions } \psi : G \longrightarrow End_K(V) \text{ vanishing at infinity} \\ \forall u, u' \in U, \;\; \psi(ugu') = \rho(u) \circ \psi(g) \circ \rho(u') \end{array} \right\}.$$

Similarly we endow $ind_U^G(\rho)$ with the norm $||f|| = \sup_{g\in G} ||f(g)||$ and we denote by $\mathcal{B}_U^G(\rho)$ the completion. We thus have

$$\mathcal{B}_U^G(\rho) := \left\{ \begin{array}{l} \text{functions } g : G \longrightarrow V \text{ vanishing at infinity} \\ \forall u \in U, \;\; f(gu) = \rho(u)^{-1}(f(g)) \end{array} \right\}.$$

Completed Hecke algebras (2/2)

Lemma (ST)

1. *$\mathcal{B}_U^G(\rho)$ is a unitary Banach representation of G.*
2. *The action of $\mathcal{H}(G,\rho)$ on ind$_U^G(\rho)$ extends to a continuous action of $\mathcal{B}(G,\rho)$ on $\mathcal{B}_U^G(\rho)$.*

Let us equip the continuous endomorphism algebra $\mathrm{End}_G^{cont}(\mathcal{B}_U^G(\rho))$ with the operator norm. Then

Proposition (ST) : *The map*

$$\begin{array}{ccc} \mathcal{B}(G,\rho) & \to & \mathrm{End}_G^{cont}(\mathcal{B}_U^G(\rho)) \\ \psi & \mapsto & \psi * - \end{array}$$

is an isometry of K-algebras.

Proof : we define the inverse isometry by the same formula as before, namely we map an endomorphism $A \in End_G(ind_U^G(\rho))$ to the function $\psi_A : g \mapsto \big(v \mapsto A(f_v)(g^{-1})\big)$.

A particular case

Here we assume that ρ extends to a (continuous) representation of G. Then the map

$$\begin{array}{rccc} \iota_\rho : & \mathcal{H}(G, 1_U) & \rightarrow & \mathcal{H}(G, \rho) \\ & \psi & \mapsto & \psi.\rho \end{array}$$

is a monomorphism of K-algebras.

Assume further that G and ρ are locally analytic and let $\mathrm{d}\rho$ be the derived action of the Lie algebra $\mathfrak{g}$ on V.

Lemma (ST)

If $\mathrm{d}\rho$ is absolutely irreducible, then ι_ρ is bijective.

Proof : We have $ind_U^G(\rho) \simeq ind_U^G(1_U) \otimes \rho$. By Frobenius reciprocity it follows that $\mathcal{H}(G, \rho) \simeq \left(ind_U^G(1_U) \otimes \rho \otimes \rho^*\right)^U$.But our hypothesis implies $\left(ind_U^G(1_U) \otimes \rho \otimes \rho^*\right)^{\mathfrak{g}=0} = ind_U^G(1_U)$ because $\mathfrak{g}$ acts trivially on $ind_U^G(1_U)$.

p-adic Satake isomorphism

Context and notation

We consider a split reductive group $\mathbf{G}$ over $\mathbb{Q}_p$. We even assume given a reductive smooth model over $\mathbb{Z}_p$, still denoted by $\mathbf{G}$.
We also need

- a Borel subgroup $\mathbf{B}$ with unipotent radical $\mathbf{N}$, over $\mathbb{Z}_p$.
- a (split) maximal torus $\mathbf{T}$ in $\mathbf{B}$.
- the root system $\Phi = \Phi(\mathbf{T}, \mathbf{G}) \subset X^*(\mathbf{T})$, its subset $\Phi^+ = \Phi(\mathbf{T}, \mathbf{B})$, and its Weyl group $W = \mathcal{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$.

We denote $\mathbb{Q}_p$-points by G, T, etc, and $\mathbb{Z}_p$-points by G_0, T_0, etc.

We will define the Satake transform which allows one to compute the Hecke algebra $\mathcal{H}(G, 1_{G_0})$.

Its original definition in general requires one to fix a square root of p in the coefficient field K, in order to have a square root $\delta^{1/2}$ of the *modulus character*

$$\begin{array}{rccc} \delta : & T & \to & K^\times \\ & t & \mapsto & p^{\mathrm{val}_p(\det(t, \mathfrak{n}))} \end{array} .$$

Classical Satake isomorphism

The Satake transform is the following map :

$$\begin{array}{rccc} S: & \mathcal{H}(G, 1_{G_0}) & \to & \mathcal{H}(T, 1_{T_0}) \\ & \psi & \mapsto & \left(t \mapsto \delta^{-\frac{1}{2}}(t) \sum_{n \in N/N_0} \psi(tn)\right) \end{array}$$

Satake's theorem asserts that S induces an isomorphism

$$\mathcal{H}(G, 1_{G_0}) \xrightarrow{\sim} \mathcal{H}(T, 1_{T_0})^W.$$

On the other hand we have $\mathcal{H}(T, 1_{T_0}) = K[T/T_0]$, and the pairing

$$\begin{array}{rcl} T \times X^*(\mathbf{T}) & \to & \mathbb{Z} \\ (t, \chi) & \mapsto & -\mathit{val}_p(\chi(t)) \end{array}$$

defines an isomorphism $T/T_0 \xrightarrow{\sim} X_* := X_*(\mathbf{T})$. Altogether we get an isomorphism

$$\mathcal{H}(G, 1_{G_0}) \xrightarrow{\sim} K[X_*]^W.$$

Remark : there is also an unnormalized version which doesn't require to fix any root of p in K.

Completion of classical Satake (1/5)

We need more precise information on the Satake isomorphism.
More notation :

- $(X_*)^+ := \{\lambda \in X_*,\ \forall \alpha \in \Phi^+,\ \langle \alpha, \lambda \rangle \geqslant 0\}$.
  - This is a fundamental domain for the action of W on X_*. We write λ^{dom} the representative of $W.\lambda$ in $(X_*)^+$.
  - We have Cartan decomposition $G = \bigsqcup_{\mu \in (X_*)^+} G_0 \mu(p^{-1}) G_0$.
- A partial order on X_* : $\lambda \preccurlyeq \mu \Leftrightarrow (\mu - \lambda \in \sum_{\alpha \in \Phi^+} \mathbb{Q}_+ \alpha^\vee)$. We always have $\lambda \preccurlyeq \lambda^{dom}$.
- $\eta := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \in X^*(\mathbf{T})_{\mathbb{Q}}$. So that $\delta^{-\frac{1}{2}}(\lambda(p^{-1})) = p^{\langle \lambda, \eta \rangle}$. We have $\lambda \preccurlyeq \mu \Rightarrow \langle \lambda, \eta \rangle \leqslant \langle \mu, \eta \rangle$.

Fact : let ψ_μ be the characteristic function of $G_0 \mu(p^{-1}) G_0$ and write $S(\psi_\mu) = \sum_\lambda p^{\langle \lambda, \eta \rangle} c(\lambda, \mu).\lambda \in K[X_*]^W$ Then $c(\lambda, \mu)$ is an integer, is 0 unless $\lambda \preccurlyeq \mu$, and is 1 if $\lambda = \mu$.

Consequence : The Satake isomorphism is isometric w.r.t. to the norm $|| \sum_{\lambda \in X_*} a_\lambda.\lambda ||_\eta := \sup_\lambda p^{\langle \eta, \lambda \rangle} |a_\lambda|$ on $K[X_*]^W$.

Completion of classical Satake (2/5)

Consider the norm $||\sum_{\lambda\in X_*} a_\lambda.\lambda||_\eta := \sup_\lambda p^{\langle\eta,\lambda^{dom}\rangle}|a_\lambda|$ on $K[X_*]$ and write $K\langle X_*\rangle_\eta$ for the corresponding completion. This norm is W-equivariant and induces the previous $||.||_\eta$ on $K[X_*]^W$. Therefore we get an isomorphism

$$\mathcal{B}(G, 1_{G_0}) \xrightarrow{\sim} K\langle X_*\rangle_\eta^W.$$

Lemma : *$K\langle X_*\rangle_\eta$ is an affinoid algebra.*

Proof : Let F^+ be a finite generating subset of the monoid $(X_*)^+$ and put $F := W.F^+$. Then the morphism of algebras

$$\begin{array}{rcl} K\langle T_\lambda\rangle_{\lambda\in F} & \to & K\langle X_*\rangle_\eta \\ T_\lambda & \mapsto & p^{\langle\eta,\lambda^{dom}\rangle}\lambda \end{array}$$

is an epimorphism of normed algebras (with Gauss norm on LHS).

Consequence : $K\langle X_*\rangle_\eta^W$, and therefore $\mathcal{B}(G, 1_{G_0})$, are affinoid algebras.

Completion of classical Satake (3/5)

Remark : when G is semisimple and adjoint, these are indeed Tate algebras. Namely, let $\lambda_1, \cdots, \lambda_r$ be the fundamental dominant coweights in X_*, the morphism

$$\begin{array}{rcl} K\langle T_1, \cdots, T_r\rangle & \to & K\langle X_*\rangle_\eta^W \\ T_i & \mapsto & p^{\langle \lambda_i, \eta\rangle} \sum_{\lambda \in W.\lambda_i} \lambda \end{array}$$

is an isometric isomorphism. Using the form of coefficients $c(\lambda, \mu)$ in the Satake transform, we deduce that $\mathcal{B}(G, 1_{G_0})$ is a Tate algebra in the variables ψ_{λ_i}, $i = 1, \cdots, r$.

Similarly, when $G = \mathrm{GL}_n$, $\mathcal{B}(G, 1_{G_0})$ is isomorphic to $K\langle T_1, \cdots, T_{n-1}, T_n, T_n^{-1}\rangle$.

In these cases the rigid spectrum of the Banach-Hecke algebra is a polydisc or a polyannulus.

Completion of classical Satake (4/5)

We now compute the rigid spectrum of $K\langle X_*\rangle_\eta$. Let $C := \widehat{\overline{K}}$ and let $\mathbf{T}' := \mathit{Spec}\,(K[X_*])$ be the dual torus of $\mathbf{T}$ over K. The valuation $\mathrm{val}_p : C^\times \longrightarrow \mathbb{Q}$ provides us with a map

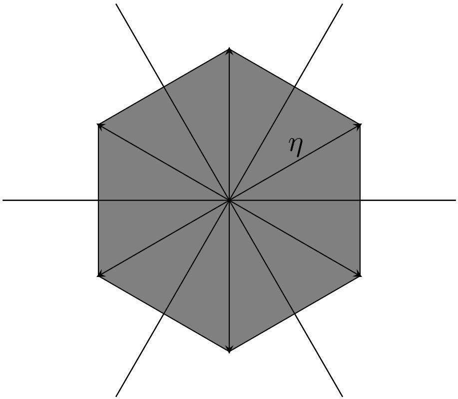
$$\mathbf{T}'(C) = X^* \otimes_{\mathbb{Z}} C^\times \xrightarrow{\mathrm{val}_p} X^*_{\mathbb{Q}}$$

Let $\mathbf{T}'_\eta := \mathit{Spm}\,(K\langle X_*\rangle_\eta)$ be the rigid spectrum of $K\langle X_*\rangle_\eta$.

Proposition : *The set* $\mathbf{T}'_\eta(C)$ is the preimage by $-\mathrm{val}_p$ of the convex hull of the orbit $W.\eta$.

Proof : Indeed an element $\zeta \in \mathbf{T}'(C)$ lies in $\mathbf{T}'_\eta(C)$ iff for all $\lambda \in X_*$ we have $|\zeta(\lambda)| \leqslant p^{\langle \lambda^{dom}, \eta\rangle}$, which amounts to $\langle -\mathrm{val}_p(\zeta), \lambda\rangle \leqslant \langle \eta, \lambda^{dom}\rangle$. Hence for any dominant $\mu \in (X_*)^+$ and any $w \in W$ we have $\langle \eta + \mathit{val}_p(\zeta^w), \mu\rangle \geqslant 0$. That these inequalities define the claimed polyhedron is an exercice in root systems.

Here is a picture of the polyhedron for SL_3.



Completion of classical Satake (5/5)

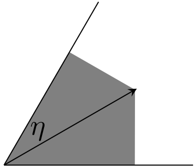
We can adapt the foregoing discussion to compute the rigid spectrum of $K\langle X_*\rangle_\eta^W$. Let $\mathbf{T}'/W := \mathit{Spec}(K[X_*]^W)$. The valuation now provides us with a map to the Weyl chamber

$$(\mathbf{T}'/W)(C) = (X^* \otimes_{\mathbb{Z}} C^\times)/W \xrightarrow{-\mathrm{val}_p} X^*_{\mathbb{Q}}/W \xrightarrow{(.)^{dom}} (X^*_{\mathbb{Q}})^+$$

Let $\mathbf{T}'_\eta/W := \mathit{Spm}(K\langle X_*\rangle_\eta^W)$ be the rigid spectrum of $K\langle X_*\rangle_\eta^W$.

Proposition : *The subset $(\mathbf{T}'_\eta/W)(C)$ of $(\mathbf{T}'/W)(C)$ is the preimage by $(-\mathrm{val}_p)^{dom}$ of $\{\chi \in (X^*_{\mathbb{Q}})^+,\ \chi \preccurlyeq \eta\}$.*

So we get the following picture :



Weighted Satake isomorphism

We now replace the trivial representation 1_{G_0} by an irreducible algebraic representation. More precisely,

- let $\xi \in (X^*)^+$ be a dominant weight, and
- let (ρ_ξ^{alg}, V_ξ) be the irreducible rational representation of $\mathbf{G}$ of highest weight ξ over $\mathbb{Q}_p$ and let $(\rho_\xi, V_{\xi,K})$ be the associated K-representation of G.

It is known that ρ_ξ^{alg} can be defined over $\mathbb{Z}_p$. Choose such a model $(\rho_{\xi,0}^{alg}, \omega_\xi)$. It provides a G_0-invariant norm $||.||$ on V_ξ, or equivalently a G_0-invariant $\mathcal{O}$-lattice $(\rho_{\xi,0}, \omega_{\xi,\mathcal{O}})$ in $V_{\xi,K}$.

Lemma : *For all $t \in T$ we have $||\rho_\xi(t)|| = |\xi(t^{dom})|$.*
Proof : The weight decomposition of V_ξ is defined on the $\mathbb{Z}_p$ model. Hence $||\rho_\xi(t)|| = \sup\{|\kappa(t)|, \kappa$ weight of T in $V_\xi\}$ The claim follows from the two facts :

- any weight κ satisfies κ^{dom} 4 ξ
- the set of weights is W-invariant.

Completion of weighted Satake (1/4)

Since the derived action $\mathrm{d}\rho_\xi$ is absolutely irreducible, the map $\iota_{\rho_\xi}; \psi \in \mathcal{H}(G, 1_{G_0}) \mapsto \psi.\rho_\xi \in \mathcal{H}(G, \rho_{\xi_{|G_0}})$ is an isomorphism of algebras. Composing its inverse with the Satake transform gives an isomorphism

$$S_\xi : \mathcal{H}(G, \rho_{\xi_{|G_0}}) \xrightarrow{\sim} K[X_*]^W.$$

Lemma : *S_ξ carries the norm $||.||$ on the LHS to the norm $||\sum_{\lambda \in X_*} a_\lambda \lambda||_{\eta+\xi} := \sup_{\lambda \in X_*} p^{\langle \eta+\xi, \lambda\rangle} |a_\lambda|$ on the RHS.*

Proof : By the previous Lemma, for any $\mu \in (X_*)^+$ we have $||\psi_\mu \rho_\xi|| = ||\rho_\xi(\mu(p^{-1}))|| = |\xi \circ \mu(p^{-1})| = p^{\langle \xi, \mu\rangle}$. Therefore the functions $\varphi_\mu := p^{\langle \xi, \mu\rangle} \psi_\mu \rho_\xi$ for $\mu \in (X_*)^+$ form an orthonormal basis of $\mathcal{H}(G, \rho_{\xi_{|G_0}})$. Hence the claim follows from the expression $S_\xi(\varphi_\mu) = \sum_{\lambda \in X_*} p^{\langle \eta+\xi, \lambda\rangle} c(\lambda, \mu)\lambda$ and the already mentioned pties of $c(\lambda, \mu)$.

Completion of weighted Satake (2/4)

Consider the norm $||\sum_{\lambda \in X_*} a_\lambda.\lambda||_{\eta+\xi} := \sup_\lambda p^{\langle \eta+\xi, \lambda^{dom} \rangle}|a_\lambda|$ on $K[X_*]$ and write $K\langle X_* \rangle_{\eta+\xi}$ for the corresponding completion. This norm is W-equivariant and induces the previous $||.||_{\eta+\xi}$ on $K[X_*]^W$. Therefore we get an isomorphism

$$\mathcal{B}(G, \rho_{\xi,0}) \xrightarrow{\sim} K\langle X_* \rangle_{\eta+\xi}^W.$$

As in the unweighted case we have :

Lemma : *$K\langle X_* \rangle_{\eta+\xi}$, $K\langle X_* \rangle_{\eta+\xi}^W$ and therefore $\mathcal{B}(G, \rho_{\xi,0})$ are affinoid algebras.*

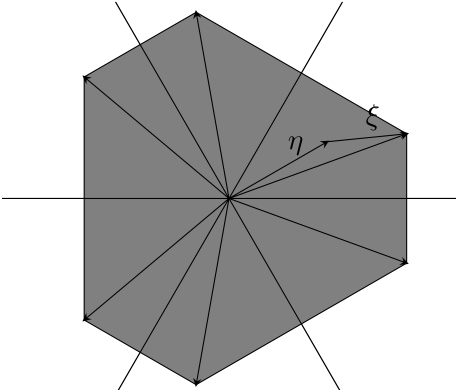
When G is semi-simple adjoint, $\mathcal{B}(G, \rho_{\xi,0})$ is again a Tate algebra, in the variables $\frac{\psi_{\lambda_i}\rho_\xi}{\langle \xi, \lambda_i \rangle}$. When $G = \mathrm{GL}_n$, the rigid spectrum $\mathcal{B}(G, \rho_{\xi,0})$ is a polyannulus.

Completion of weighted Satake (3/4)

We may also compute the C-valued points of the rigid spectrum $\mathbf{T}'_{\eta+\xi} = \mathit{Spm}\,(K\langle X_*\rangle_{\eta+\xi})$ as in the unweighted case.

Proposition : *The set $\mathbf{T}'_{\eta+\xi}(C)$ is the preimage by $-\mathrm{val}_p$ of the convex hull of the orbit $W.(\eta+\xi)$.*

Example : for SL_3.



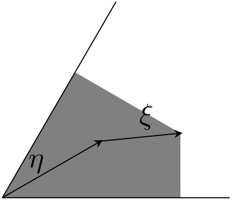
Completion of weighted Satake (4/4)

Similarly we compute the C-valued points of the rigid spectrum $\mathbf{T}'_{\eta+\xi}/W = \mathit{Spm}\,(K\langle X_*\rangle^W_{\eta+\xi})$.

Proposition : The set $(\mathbf{T}'_{\eta+\xi}/W)(C)$ is the preimage of the set $\{\chi \in (X^*_{\mathbb{Q}})^+,\ \chi$ 4 $\eta+\xi\}$ by the map

$$(-\mathrm{val}_p)^{dom} : (\mathbf{T}'_{\eta+\xi}/W)(C) \longrightarrow (X^*_{\mathbb{Q}})^+.$$

For SL_3 we get the picture :



Specialization of the universal module

Classical case : The universal module is the (smooth) induced representation $\mathit{ind}_{G_0}^{G}(1_{G_0})$. Given a K-valued character ζ of the Hecke algebra $\mathcal{H}(G, 1_{G_0})$, we can specialize the universal module :

$$H_{1,\zeta} := \mathit{ind}_{G_0}^{G}(1_{G_0}) \otimes_{\mathcal{H}(G,1_{G_0}),\zeta} K.$$

It is known that $H_{1,\zeta}$ has finite length, has a unique G-irred. quotient $V_{1,\zeta}$, and is generically irreducible.

Weighted case : let ξ be a dominant weight, and ρ_ξ the ass. irred. rational repr. Here the universal module is $\mathit{ind}_{G_0}^{G}(\rho_{\xi_{|G_0}})$ and we may specialize it w.r.t a character ζ of $\mathcal{H}(G, \rho_{\xi_{|G_0}})$. We get $H_{\xi,\zeta} := \mathit{ind}_{G_0}^{G}(\rho_{\xi_{|G_0}}) \otimes_{\mathcal{H}(G,\rho_{\xi_{|G_0}}),\zeta} K$. The natural G-equivariant isomorphism $\mathit{ind}_{G_0}^{G}(1_{G_0}) \otimes \rho_\xi \xrightarrow{\sim} \mathit{ind}_{G_0}^{G}(\rho_{\xi_{|G_0}})$ is Hecke equivariant w.r.t. $\iota_{\rho_\xi} : \mathcal{H}(G, 1_{G_0}) \xrightarrow{\sim} \mathcal{H}(G, \rho_{\xi_{|G_0}})$, and therefore induces an isomorphism $H_{1,\iota^*_{\rho_\xi}(\zeta)} \otimes \rho_\xi \xrightarrow{\sim} H_{\xi,\zeta}$.

Specialization of the completed universal module

As before we fix $\rho_{\xi,0}$ a model of $\rho_{\xi_{|G_0}}$ over $\mathbb{Z}_p$. The completed universal module is $\mathcal{B}_{G_0}^G(\rho_{\xi,0})$ and we may specialize it w.r.t to any continuous character ζ of $\mathcal{B}(G,\rho_{\xi,0})$. We then get a Banach unitary representation of G :

$$B_{\xi,\zeta} := \mathcal{B}_{G_0}^G(\rho_{\xi,0})\widehat{\otimes}_{\mathcal{B}(G,\rho_{\xi,0}),\zeta}K.$$

We may also view $B_{\xi,\zeta}$ as the completion of the loc. alg. finite length repr. $H_{\xi,\zeta}$ w.r.t. to the quotient semi-norm coming from $ind_{G_0}^G(\rho_{\xi_{|G_0}})$. Note that this semi-norm is defined by the image of the (finitely $\mathcal{O}G$-generated) $\mathcal{O}$-lattice $ind_{G_0}^G(\rho_{\xi,0})$.

In general $\mathcal{B}_{\xi,\zeta}$ is not expected to be of finite length, nor even admissible.
One big open problem in the theory is whether $B_{\xi,\zeta}$ is non-zero. More precisely, Breuil, Schneider and Teitelbaum make the following conjecture.

A representation theoretic Conjecture

Conjecture : *Let* $\zeta \in \mathbf{T}'(K)/W$. *Then the locally algebraic repr.* $H_{\xi,\zeta}$ *admits an invariant lattice if and only if* $\zeta \in \mathbf{T}'_{\eta+\xi}(K)/W$.

Remarks : The "only if" part is now straightforward. Indeed if $H_{\xi,\zeta}$ admits an invariant lattice, then the image of the lattice $ind_{G_0}^{G}(\rho_{\xi,0})$ is itself a lattice. This implies that the character ζ is continuous for the norm associated to this lattice, therefore extends to a continuous character of $\mathcal{B}(G,\rho_{\xi,0})$.

Here are a few cases where the "if" part is known :

- for $G = \mathrm{GL}_2$, this is contained in a paper by Berger and Breuil and uses (φ,Γ) modules following Colmez's strategy.
- for $G = \mathrm{GL}_3$ and $\xi = 0$, this follows from the freeness of $ind_{G_0}^{G}(1_{G_0})$ as a module over $\mathcal{H}(G,1_{G_0})$, due to Bellaiche and Otwinowska.
- for $G = \mathrm{GL}_n$, $\xi = 0$ and ζ regular, lifting to $\tilde{\zeta}$ such that $\delta^{-1/2}\tilde{\zeta}$ is integrally valued. In this case we even have an isomorphism $\mathcal{B}_{1,\zeta} \xrightarrow{\sim} Ind_B^G(\delta^{-1/2}\tilde{\zeta})^c$.

Classical unramified functoriality (1/3)

$\Gamma_{\mathbb{Q}_p} := \mathit{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $I_{\mathbb{Q}_p}$ = inertia, $W_{\mathbb{Q}_p} \subset \Gamma_{\mathbb{Q}_p}$ = Weil group. Recall that a smooth K-valued representation (ρ, V) of $W_{\mathbb{Q}_p}$ is called *unramified* if $\rho_{|I_{\mathbb{Q}_p}}$ is trivial and $\rho(\mathrm{Frob}_p)$ is semi-simple.

Assume $V = K^n$. Let $\mathbf{T}'$ be the torus of diag matrices in $\mathrm{GL}_{n,K}$, with Weyl group W. The unramified Langlands correspondence is given by

$$\begin{aligned} & \{\text{isom. class. of } n\text{-dim unram. } K\text{-repr. of } W_{\mathbb{Q}_p}\} \\ \longleftrightarrow \quad & \{\text{cgcy. class. of s/simple elts in } \mathrm{GL}_n(K)\} = (\mathbf{T}'/W)(K) \\ \overset{\zeta \mapsto V_{1,\zeta}}{\longleftrightarrow} \quad & \{\text{isom. cl. smth. } \mathrm{GL}_n(\mathbb{Z}_p)\text{-spherical } K\text{-repr. of } \mathrm{GL}_n(\mathbb{Q}_p)\} \end{aligned}$$

Remark : $\mathrm{GL}_n(\mathbb{Z}_p)$-spherical K-repr. of $\mathrm{GL}_n(\mathbb{Q}_p)$ are also called “unramified”.

Classical unramified functoriality (2/3)

More generally, start with a split group $\mathbf{G}$ over $\mathbb{Q}_p$ and let $\mathbf{G}'$ be its dual split group, over K. In terms of root data, just exchange the roles of X_* and X^*, roots and coroots.
An *unramified K-parameter* for $\mathbf{G}$ is a locally constant group homomorphism $W_{\mathbb{Q}_p} \longrightarrow \mathbf{G}'(K)$ trivial on $I_{\mathbb{Q}_p}$ and taking a Frobenius to a semi-simple element.Assume given a reductive model $\mathbf{G}$ over $\mathbb{Z}_p$. The unramified Langlands correspondence for $\mathbf{G}$ is

$$\begin{array}{cl} & \{\text{isom. class. of unram. } K\text{-parameters } W_{\mathbb{Q}_p} \longrightarrow \mathbf{G}'(K)\} \\ \longleftrightarrow & \{\text{cgcy. class. of s/simple elts in } \mathbf{G}'(K)\} = (\mathbf{T}'/W)(K) \\ \overset{\zeta \mapsto V_{1,\zeta}}{\longleftrightarrow} & \{\text{isom. cl. of smth } \mathbf{G}(\mathbb{Z}_p)\text{-spherical } K\text{-repr. of } \mathbf{G}(\mathbb{Q}_p)\} \\ \longleftrightarrow & \{\text{unramified } L\text{-paquets for } \mathbf{G}\} \end{array}$$

Classical unramified functoriality (3/3)

Now let $\mathbf{G}_1$, $\mathbf{G}_2$ be two split groups over $\mathbb{Q}_p$ and let $\mathbf{G}'_1 \xrightarrow{\varphi} \mathbf{G}'_2$ be a morphism of the dual groups. The principle of functoriality is summarized by the following diagram.

$$\begin{array}{ccc}
\{\text{unr. param. } W_{\mathbb{Q}_p} \longrightarrow \mathbf{G}'_1(K)\} & \stackrel{LC}{\longleftrightarrow} & \{\text{unr. } L\text{-paquets for } \mathbf{G}_1\} \\
\big\downarrow \varphi\circ - & & \vdots \\
\{\text{unr. param. } W_{\mathbb{Q}_p} \longrightarrow \mathbf{G}'_2(K)\} & \stackrel{LC}{\longleftrightarrow} & \{\text{unr. } L\text{-paquets for } \mathbf{G}_2\}
\end{array}$$

Remark : if one erases the "unr.", the bijections LC and the dotted vertical maps are still conjectural in general.

A possible generalization of this diagram to the *continuous* theory (as opposed to the *smooth* one) should involve the previously constructed Banach representations $B_{\xi,\zeta}$ and the so-called *crystalline* representations.

Crystalline representations and filtered modules (1/2)

Crystalline repr are more easily studied through their associated filtered φ-module. Fontaine has defined an additive functor D_{cris} from $\mathrm{Rep}_K^c(\Gamma_{\mathbb{Q}_p})$ to the category $\mathrm{MF}_K^{\mathbb{Z}}$ of "$\mathbb{Z}$-filtered φ-module over $\mathbb{Q}_p$ with coefficients in K"

Theorem (Colmez-Fontaine) : *D_{cris} induces an equivalence of categories from $\mathrm{Rep}_K^{crys}(\Gamma_{\mathbb{Q}_p})$ to the category $\mathrm{MF}_K^{\mathbb{Z},a}$ of "admissible" $\mathbb{Z}$-filtered φ-modules.*

Crystalline representations and filtered modules (2/2)

Definition : A filtered φ-module over $\mathbb{Q}_p$ with coefficients in K is a triple $(D, \varphi, \mathrm{Fil}^\bullet D)$ where

- D is a finite dimensional vec. sp. over K.
- φ is a K-linear automorphism of D.
- $\mathrm{Fil}^\bullet D$ is a decreasing exhaustive separated $\mathbb{Q}$-filtration of D.

The Newton number is $t_N(D) := \mathrm{val}_p(\det(\varphi))$.
The Hodge number is $t_H(D) := \sum_{i \in \mathbb{Q}} i \, dim_K(\mathrm{Fil}^i D / \mathrm{Fil}^{i+1} D)$.

Definition : A filtered φ-module is called *admissible* if $t_H(D) = t_N(D)$ and $t_H(D') \leqslant t_N(D')$ for all φ-submodule D'.

The category MF_K^a of admissible filtered modules is abelian, and even Tannakian, and D_{cris} is in fact a $\otimes$-functor.

The functor WD is given by $(D, \varphi, \mathrm{Fil}) \mapsto (D, \varphi^{ss})$.

Interlude on dominance order and polygons

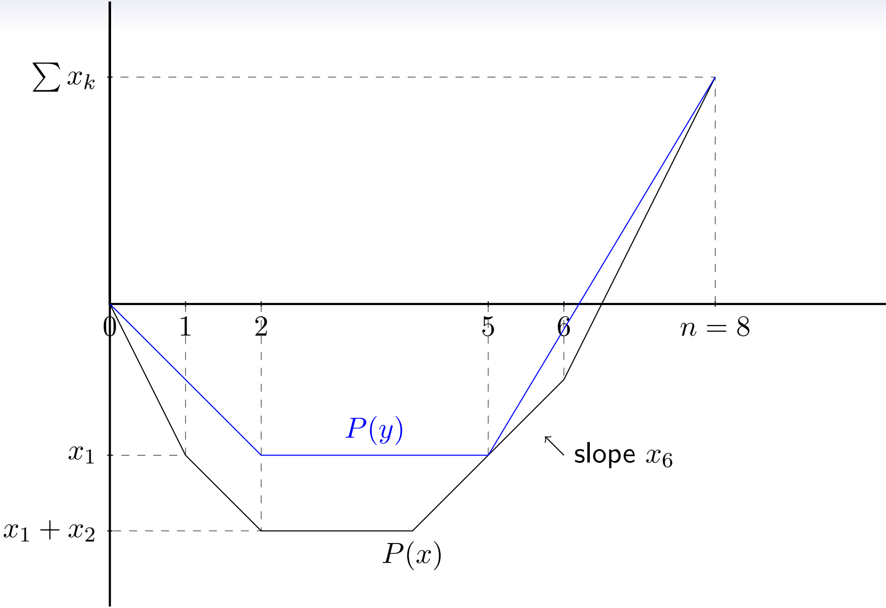
We recall different expressions of the dominance order for GL_n. Let $\mathbf{T}'$ be the diagonal torus and $\mathbf{B}'$ the lower triangular Borel sgp. In the natural identification $X_*(\mathbf{T}') = \mathbb{Z}^n$, the Weyl chamber associated to B is

$$(X_*(\mathbf{T}')_{\mathbb{R}})^+ = (\mathbb{R}^n)_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n,\ x_1 \leqslant \cdots \leqslant x_n\}$$

The dominance order on this Weyl chamber is given by

$$y \preccurlyeq x \Longleftrightarrow \left(\forall i = 1, \cdots, n,\ \sum_{k=1}^{i} x_k \leqslant \sum_{k=1}^{i} y_k \text{ and } \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k \right)$$

It is customary to represent an element $x \in (\mathbb{R}^n)^+$ by a “polygon” $P(x)$, namely the graph of the piecewise linear function $[0, n] \longrightarrow \mathbb{R}$ which interpolates the origin and the points $(i, \sum_{k=1}^{i} x_k)$ for $i = 1, \cdots, n$. In this language $y \preccurlyeq x$ iff $P(x)$ “lies below” $P(y)$ with the same end-points.


$\sum x_k$
x_1
$x_1 + x_2$
0
1
2
5
6
$n = 8$
$P(y)$
$P(x)$
slope x_6

Polygons of a filtered module

Let $D = K^n$. As before $\mathbf{T}' \subset \mathbf{B}'$ is the lower Borel pair in $GL_{n,K}$, and W is the Weyl group.

Newton polygon : We have the Chevalley map

$$\begin{array}{rcl} \mathrm{GL}_n(K) & \rightarrow & (\mathbf{T}'/W)(K) \\ \varphi & \mapsto & \zeta_\varphi = \text{ conj. cl. of. } \varphi^{ss} \end{array}.$$

Composing with $(\mathrm{val}_p)^{dom} : (\mathbf{T}'/W)(K) \longrightarrow X^*_{\mathbb{Q}}/W \xrightarrow{\sim} (X^*_{\mathbb{Q}})^+$, we get a an element $\mathrm{val_p}(\zeta_\varphi)^{dom}$ in the Weyl chamber. Its polygon is the *Newton polygon* of the characteristic polynomial of φ.

Hodge polygon : On the other hand, we have the map

$$\begin{array}{rcl} \{\text{filtr. } \mathrm{Fil}^\bullet \text{ on } D\} & \rightarrow & (X^*_{\mathbb{Q}})^+ = (\mathbb{Q}^n)^+ \\ \mathrm{Fil}^\bullet & \mapsto & \kappa_{\mathrm{Fil}^\bullet} = \text{sequence of breaks of } \mathrm{Fil}^\bullet \end{array}$$

Its polygon is called the *Hodge polygon*.

Around the admissibility condition

Lemma : *If the filtered module* $(D, \varphi, \mathrm{Fil}^\bullet)$ *is admissible, then we have* $\mathrm{val}_p(\zeta_\varphi)^{dom} \preccurlyeq \kappa_{\mathrm{Fil}^\bullet}$*, i.e. the Newton polygon lies above the Hodge polygon with same end-points.*

Proof : Let $\mathrm{val}_p(\zeta_\varphi)^{dom} = (v_1 \leqslant \cdots \leqslant v_n)$ and let D_i be the direct summand φ-submodule where eigenvalues of φ all have valuation $\leqslant v_i$. We have $t_N(D_i) = \sum_{k=1}^{i} v_k$, whereas $t_H(D_i) \geqslant \sum_{k=1}^{i} \kappa_k$. Varying i and using $t_H(D_i) \leqslant t_N(D_i)$ we get the desired inequality.

Proposition : *Let* $D = K^n$*. Let* $\zeta \in (\mathbf{T}'/W)(K)$ *and* $\kappa \in (\mathbb{Q}^n)^+$ *such that* $\mathrm{val}_p(\zeta)^{dom} \preccurlyeq \kappa$*, Then there exists*

- $\varphi \in \mathrm{GL}_n(K)$ *with* $\zeta = \zeta_\varphi$*, and*
- *a filtration* $\mathrm{Fil}^\bullet$ *with* $\kappa_{\mathrm{Fil}^\bullet} = \kappa$*,*

such that $(D, \varphi, \mathrm{Fil}^\bullet)$ *is admissible.*

Proof of the Proposition : The filtrations of type κ are parametrized by a flag variety $\mathcal{F}_\kappa$. For a fixed vec. subsp. $D' \subset D$, the map $\mathrm{Fil}^\bullet \mapsto t_H(D')$ is upper semi-continuous on $\mathcal{F}_\kappa$ since we have $t_H(D') = \sum_{i=1}^n (\kappa_{i+1} - \kappa_i)\, \mathit{dim}(\mathrm{Fil}^{\kappa_i} D \cap D') + \kappa_1\, \mathit{dim}(D')$. Therefore for any integer t, the set $\mathcal{F}_\kappa^{D',t} := \{\mathrm{Fil}^\bullet \in \mathcal{F}_\kappa,\ t_H(D') \leqslant t\}$ is open. Moreover it is non-empty iff the point $(\mathit{dim}_K D', t)$ lies above the polygon of κ.

Now we choose a *regular* element $\varphi \in \mathrm{GL}_n(K)$ in the conjugacy class ζ (note that φ need not be semi-simple). Then there are only *finitely many* φ-submodules D'. For such a φ-submodule, the point $(\mathit{dim}(D'), t_N(D'))$ lies on the Newton polygon, hence above the polygon of κ. Therefore, $(D, \varphi, \mathrm{Fil}^\bullet)$ is admissible for a generic choice of the filtration $\mathrm{Fil}^\bullet$ of type κ.

Remark : It need not be possible to choose φ semi-simple : Drinfeld's upper half space has no rational points...

Conjectural coarse crystalline correspondence (1/2)

Keep the notation $\zeta \in (\mathbf{T}'/W)(K)$ and fix a dominant weight $\xi \in (X^*)^+$. Then consider the properties :

1. The Banach K-representation $B_{\xi,\zeta}$ of $\mathrm{GL}_n(\mathbb{Q}_p)$ admits a GL_n-invariant norm.
2. There is a "Frobenius" $\varphi \in \mathrm{GL}_n(K)$ such that $\zeta_\varphi = \zeta$ and a filtration $\mathrm{Fil}^\bullet K^n$ of type $\kappa = (-(\eta + \xi))^{dom}$, such that $(K^n, \varphi, \mathrm{Fil}^\bullet)$ is admissible.

Corollary : *Property 1 implies Property 2*

Indeed 1 implies $(-\mathit{val}_p(\zeta))^{dom}$ 4 $\eta + \xi$, which amounts to $\mathit{val}_p(\zeta)^{dom}$ 4 $(-(\eta + \xi))^{dom}$ which was just seen as equivalent to 2.

Moreover, previous repr. theoretic conjecture is equivalent to :

Conjecture : *Properties 1 and 2 are equivalent.*

Assuming this conjecture the next question would be :

Question : *Is there a correspondence between equiv. classes of invariant norms on $B_{\xi,\zeta}$ and admissible $(K^n, \varphi, \mathrm{Fil}_\bullet)$ with $\zeta_\varphi = \zeta$ and $\kappa_{\mathrm{Fil}^\bullet} = (-(\eta + \xi)^{dom})$?*

Conjectural coarse crystalline correspondence (2/2)

If true, one would like to see the Conjecture as a coarse crystalline correspondence. However, this rises the following

Problem : Crystalline repr. are associated to $\mathbb{Z}$-filtered modules. But in general, $\eta \in \frac{1}{2}\mathbb{Z}$. Indeed, in the identification $X^*(\mathbf{T}) = \mathbb{Z}^n$ we have

$$\begin{aligned} \eta &= \left(-\frac{d-1}{2}, -\frac{d-3}{2}, \dots, \frac{d-1}{2}\right) \\ &= (0, 1, \cdots, d-1) - \frac{d-1}{2}(1, 1, \cdots, 1) = \tilde{\eta} - \frac{d-1}{2}\det \end{aligned}$$

Recall Geometry realizes "Hecke normalization" of Langlands cor. $\pi_H(\sigma) = \pi_L(\sigma) \otimes |\det|^{(d-1)/2}$. Therefore, natural to renormalize

$$B_{\xi,\zeta} \longleftrightarrow (\varphi, \mathrm{Fil}_\bullet) \qquad \text{with } \zeta_\varphi = \zeta.(p, \cdots, p)^{(d-1)/2}$$

$$\text{and } \kappa_{\mathrm{Fil}^\bullet} = (-(\tilde{\eta} + \xi))^{dom}$$

Natural questions arising

1. Can this be generalized to other (split) groups ?
2. Can this be generalized to other smooth irred representations of $\mathrm{GL}_n(\mathbb{Q}_p)$?
3. Can this be generalized to finite extensions of $\mathbb{Q}_p$?

Conjectural crystalline functoriality (1/4)

This uses Tannakian formalism, following Kottwitz and Rapoport.

Let $\mathbf{G}$ be split reductive over $\mathbb{Q}_p$ and $\mathbf{G}'$ its dual over K. Start with a pair (b, ν) where :

- $b \in \mathbf{G}'(K)$
- $\nu : \mathbb{D} \longrightarrow G'$ with the $\mathbb{D}$ the diagonalizable group over K such that $X^*(\mathbb{D}) = \mathbb{Q}$.

Such a pair defines a functor :

$$\begin{array}{llcl} D_{b,\nu} & \mathrm{Rep}_K(\mathbf{G}') & \to & \mathrm{MF}_K \\ & (V, \rho) & \mapsto & (V, \rho(b), \mathrm{Fil}^\bullet_\nu V) \end{array}$$

where $\mathrm{Fil}^x_\nu V = \sum_{y \geqslant x} V_y$ with V_y the eigensubspace of weight y w.r.t the cocharacter $\rho \circ \nu$.

The pair (b, ν) is called admissible if $D_{b,\nu}$ factors through MF^a_K.

If moreover ν factors through $\mathbb{D} \longrightarrow \mathbb{G}_m$, then by Colmez-Fontaine, we get a functor $\mathrm{Rep}_K(\mathbf{G}') \longrightarrow \mathrm{Rep}^{crys}_K(\Gamma_{\mathbb{Q}_p})$, whence a “crystalline parameter” $\Gamma_{\mathbb{Q}_p} \longrightarrow \mathbf{G}'(K')$ for some finite extension K' of K.

Conjectural crystalline functoriality (2/4)

Fix Borel pair $(\mathbf{T}' \subset \mathbf{B}')$ in $\mathbf{G}'$ and let W be the Weyl group.
Analog of Newton polygon : Composing the Chevalley map

$$\begin{array}{rcl} \mathbf{G}'(K) & \to & (\mathbf{T}'/W)(K) \\ b & \mapsto & \zeta_b = \text{ conj. cl. of. } b^{ss} \end{array}$$

with $(\mathrm{val}_p)^{dom} : (\mathbf{T}'/W)(K) \longrightarrow X^*_{\mathbb{Q}}/W \xrightarrow{\sim} (X^*_{\mathbb{Q}})^+$, we get an element $\mathrm{val_p}(\zeta_\varphi)^{dom}$ in the Weyl chamber $(X^*)^+_{\mathbb{Q}}$.

Analog of Hodge polygon : The $G(K)$-conjugacy class of ν gives an element of $X_*(\mathbf{T}')_{\mathbb{Q}}/W$ whence an element κ_ν of the Weyl chamber, via the bijection $(.)^{dom}$.

Proposition : *Assume b regular. Then there is ν such that (b, ν) is admissible iff we have val$_p(\zeta_b)^{dom} \preccurlyeq \kappa_\nu$.*

Conjectural crystalline functoriality (3/4)

Keep $\zeta \in (\mathbf{T}'/W)(K)$, fix $\xi \in (X^*)^+$ and consider the properties :

1. The Banach K-representation $B_{\xi,\zeta}$ of $\mathbf{G}(\mathbb{Q}_p)$ admits a $\mathbf{G}(\mathbb{Q}_p)$-invariant norm.
2. There exist a $\mathbb{Q}$-cocharacter ν of type $\kappa_\nu = (-(\eta+\xi))^{dom}$ such that the pair (b_ζ, ν) for $\mathbf{G}'$ is admissible.

Corollary : *Property 1 implies Property 2*

The previous repr. theoretical conjecture is equivalent to :
Conjecture : *Properties 1 and 2 are equivalent.*

Assuming this conjecture the next question would be :
Question : *Is there a correspondence between equiv. classes of invariant norms on $B_{\xi,\zeta}$ and admissible pairs (b, ν) with $\zeta_b = \zeta$ and $\kappa_\nu = (-(\eta+\xi)^{dom})$?*

We note that pairs (b, ν) are functorial in $\mathbf{G}'$.

Conjectural crystalline functoriality (4/4)

Problem : A priori $\eta \in \frac{1}{2}X^*(\mathbf{T})$.

Remark : When $\mathbf{G}$ is semisimple simply connected, $\eta \in X^*(\mathbf{T})$.

In general, Breuil and Schneider elaborate on Fontaine's theory to interpret admissible $\frac{1}{2}\mathbb{Z}$-filtered φ-modules as certain representations of an extension of $\Gamma_{\mathbb{Q}_p}$ by $\mathbb{Z}/2\mathbb{Z}$.