Continuous representation theory of p-adic Lie groups

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Basics on p-adic Lie groups.

Definition : a p-adic Lie group G is a group object in the category of locally $\mathbb{Q}_p$-analytic manifolds.

Locally $\mathbb{Q}_p$-analytic manifolds are defined in the same way as differentiable $\mathbb{R}$-manifolds. Smooth vector-valued functions on $\mathbb{R}^n$ are replaced by locally analytic vector-valued functions on $\mathbb{Z}_p^n$, *i.e.* functions that are locally given by convergent power series.

Straightforward properties :

- G is a totally disconnected locally compact topological group. In particular, the unit element has a basis of neighborhoods consisting of open compact subgroups.
- G has a Lie algebra $\mathfrak{g}$ over $\mathbb{Q}_p$.

Rk : one may replace $\mathbb{Q}_p$ by a finite extension L.

Nice open pro-p-subgroups

Let G be a d-dimensional p-adic Lie group. Start with a local chart $\psi : \mathbb{Z}_p^d \hookrightarrow G$ with image an open neighborhood of $\psi(0) = e_G$.

By definition there are :

- a power series $f(x,y) = \sum_{\alpha,\beta\in\mathbb{N}^d} a_{\alpha,\beta} x^\alpha y^\beta$ with $a_{\alpha,\beta} \in \mathbb{Q}_p^d$,
- an open neighborhood $p^h\mathbb{Z}_p^d$ of 0 in $\mathbb{Z}_p^d$, such that
- f converges to an analytic funtion $p^h\mathbb{Z}_p^d \times p^h\mathbb{Z}_p^d \longrightarrow \mathbb{Z}_p^d$ and $\psi(f(x,y)) = \psi(x)\psi(y)^{-1}$.

Note that $a_{0,0} = 0$ and that the linear part of f is $x - y$.

For higher degree terms, convergence of $f(p^h, p^h)$ tells us that $(\alpha,\beta) \mapsto |a_{\alpha,\beta} p^{h(|\alpha|+|\beta|)}|$ cvges to 0. Since replacing ψ by $\psi \circ p^{h'}$ multiplies $a_{\alpha,\beta}$ by $p^{h'(|\alpha|+|\beta|-1)}$, we thus may arrange for

- $a_{\alpha,\beta}$ to lie in $p\mathbb{Z}_p^d$ for all α,β with $|\alpha| + |\beta| > 1$, and
- f to converge on the whole $\mathbb{Z}_p^d \times \mathbb{Z}_p^d$. (*i.e.* $h = 0$).

Nice open pro-p-subgroups (cont'd)

In this case, the image $H = \psi(\mathbb{Z}_p^d)$ of our chart is an open compact subgroup of G. More generally for any n, the subset $H_n := \psi(p^n\mathbb{Z}_p^d)$ is an open subgroup of H, and the collection of all H_n's is a basis of neighborhoods of e_G.

Note that H_n is normal in H and ψ induces a group isomorphism $(p^n\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p)^d \xrightarrow{\sim} H_n/H_{n+1}$. In particular H is pro-p.

The profinite group H is an example of a uniform pro-p-group. The same group, equiped with the decreasing filtration $(H_n)_{n\in\mathbb{N}}$, is an example of a p-valued group, in the sense of Lazard.

Definition : a topologically finitely generated pro-p-group G with lower p-series $(P_i(G))_{i\in\mathbb{N}}$ (inductively defined by $P_{i+1}(G) := P_i(G)^p[P_i(G), G]$) is called uniform if $[G, G] \subset G^p$ and the index $|P_i(G)/P_{i+1}(G)|$ is independent of i.

p-valued groups

A p-valuation on a group G is a map $\omega : G \longrightarrow [1/(p-1), +\infty]$ such that $\omega^{-1}(+\infty) = \{e_G\}$ and

1. $\omega(gh^{-1}) \geqslant \min(\omega(g), \omega(h))$
2. $\omega(ghg^{-1}h^{-1}) \geqslant \omega(g) + \omega(h)$
3. $\omega(g^p) = \omega(g) + 1$

It defines a non-increasing $\mathbb{R}$-filtration

$$x \mapsto G_x := \{g \in G, \omega(g) \geqslant x\}$$

whose graduate $\mathrm{gr}^{\omega}_{\bullet}(\mathrm{G})$ is a graded torsion-free $\mathbb{F}_p[\pi]$ Lie algebra, with bracket induced by commutators and π-action induced by the p^{th} power map in G.

On a uniform group G, the integrally valued function

$$g \mapsto \omega(g) := \sup\{i \in \mathbb{N}, g \in P_i(G)\}$$

is a p-valuation. For our home-made open s/g H, the Lie algebra $\mathrm{gr}^{\omega}_{\bullet}(\mathrm{H})$ is commutative (all brackets vanish).

Coordinates on p-valued groups

Let (G, ω) be a complete p-valued group. Assume that $\mathrm{gr}^{\omega}_{\bullet}(\mathrm{G})$ has finite rank over $\mathbb{F}_p[\pi]$ and let $(g_i)_{i=1,\cdots,d}$ be representatives in G of a basis of $\mathrm{gr}^{\omega}_{\bullet}(\mathrm{G})$ over $\mathbb{F}_p[\pi]$ (in the graded sense).

Theorem (Lazard) : *The map*

$$\begin{array}{rccc} \Psi : & \mathbb{Z}_p^d & \longrightarrow & G \\ & (x_1, \cdots, x_d) & \longmapsto & g_i^{x_i} \cdot g_2^{x_2} \cdots g_d^{x_d} \end{array}$$

is a global analytic chart for G.

Moreover we have

$$\omega(g_i^{x_i} \cdots g_d^{x_d}) = \inf_{i=1,\cdots,d} (1 + \mathsf{val}_p(x_i))$$

An abstract group is called p-valuable if it admits a p-valuation for which it is complete with graduate of finite rank over $\mathbb{F}_p[\pi]$.

Characterization of compact p-adic Lie groups

Theorem (Lazard)

A t.d. locally compact group admits a locally $\mathbb{Q}_p$-analytic structure if and only if it admits a p-valuable open pro-p-subgroup. Moreover this analytic structure is unique.

Consequences :

- Products of analytic groups are analytic.
- Any abstract group morphism between analytic groups is analytic.
- Closed subgroups of analytic groups are analytic.

Indeed all these statements can be checked most easily on p-valuable groups.

Let G be a profinite group. The Iwasawa algebra of G is the completed group ring

$$\mathbb{Z}_p[[G]] := \varprojlim_H \mathbb{Z}_p[G/H]$$

where H runs over open subgroups of G. It is a compact $\mathbb{Z}_p$-algebra.

Example : When $G = \mathbb{Z}_p$, this algebra is isomorphic to $\mathbb{Z}_p[[T]]$ and was studied in detail by Iwasawa.

Elementary properties :

- The canonical map $\mathbb{Z}_p[G] \longrightarrow \mathbb{Z}_p[[G]]$ is injective and has dense image. Its restriction to G is a homeo onto its image.
- If G is pro-p then $\mathbb{Z}_p[[G]]$ is local with radical $R := (p) + I$ (I augm. ideal).
- The R-adic tgy is finer than the inverse limit tgy with equality if G has finite type.

The following result is a major tool in the study of Iwasawa algebras and more general distributions algebras for p-adic Lie groups. Let (G, ω) be a p-valued complete group of finite rank and define for $x \in \mathbb{R}_+$:

$$\mathbb{Z}_p[G]_x := \begin{array}{l}\text{submod. gen by elements } p^h(1-g_1)\cdots(1-g_k) \\ \text{with } h+\omega(g_1)+\cdots\omega(g_k) \geqslant x\end{array}$$

This is a non-increasing $\mathbb{R}_+$-filtration of $\mathbb{Z}_p[G]$, in the sense that $\mathbb{Z}_p[G]_0 = \mathbb{Z}_p[G]$ and $\mathbb{Z}_p[G]_x \mathbb{Z}_p[G]_y \subset \mathbb{Z}_p[G]_{x+y}$ for all x, y.

The induced filtration on $\mathbb{Z}_p$ is the natural one, so that $\mathrm{gr}^\omega_\bullet(\mathrm{Z_p[G]})$ is a $\mathbb{F}_p[\pi] = \mathrm{gr}_\bullet(\mathrm{Z_p})$-algebra.

Remark : One checks that the map

$$f \in \mathbb{Z}_p[G] \mapsto ||f||_{1/p} := p^{-\inf\{x, f \in \mathbb{Z}[G]_x\}}$$

is a multiplicative norm on $\mathbb{Z}_p[G]$.

Theorem (Lazard)

1. *The completion of* $\mathbb{Z}_p[G]$ *for the above filtration (or for the norm* $||.||_\omega$*) identifies with* $\mathbb{Z}_p[[G]]$.
2. *There is a canonical isomorphism of graded* $\mathbb{F}_p[\pi]$*-algebras*

$$U(\mathrm{gr}^\omega_\bullet(\mathrm{G})) \xrightarrow{\sim} \mathrm{gr}^\omega_\bullet(\mathbb{Z}_\mathrm{p}[\mathrm{G}])$$

Here U denotes the enveloping algebra.

In the case of our nice open cpct sgp H defined above, the graded Lie algebra $\mathrm{gr}_\bullet(\mathrm{H})$ is commutative, so we get

$$\mathrm{Sym}_{\mathbb{F}_p[\pi]}(\mathrm{gr}_\bullet\mathrm{H}) \simeq \mathbb{F}_\mathrm{p}[\pi][\mathrm{T}_1, \cdots, \mathrm{T}_\mathrm{d}] \xrightarrow{\sim} \mathrm{gr}_\bullet(\mathbb{Z}_\mathrm{p}[[\mathrm{H}]]).$$

This result is fundamental in the study of Iwasawa algebras since it allows one to use techniques from filtered ring theory.

Filtered rings techniques (1/2)

Let R be a ring and let $F_\bullet R$ be a decreasing exhaustive and separated $\mathbb{Z}$-filtration of R. Assume that R is complete for the filtration $F_\bullet$ in the sense that $R = \varprojlim_n R/F_nR$. The first useful properties are :

- If $\mathrm{gr}_\bullet \mathrm{R}$ is a (left or right) noetherian ring, so is R.
- If $\mathrm{gr}_\bullet \mathrm{R}$ has no zero divisor, then R has the same property.

Applied to the Iwasawa algebra of a compact p-adic Lie group, this gives

- $\mathbb{Z}_p[[G]]$ is left and right noetherian.
- If G is p-valuable, $\mathbb{Z}_p[[G]]$ has no zero divisors.

Assume further on that R is noetherian. The filtration $F_\bullet$ is then an example of a Zariskian filtration. For such filtrations, many homological properties can be lifted from the associate graded ring. The following will be used in Teitelbaum's second lecture :

- If $R \longrightarrow S$ is a morphism of Zariskian filtered rings such that $\mathrm{gr}_\bullet \mathrm{S}$ is flat over $\mathrm{gr}_\bullet \mathrm{R}$, then S is flat over R.

Filtered rings techniques (2/2)

If M is a module on a ring R, one defines the grade of M by

$$j_R(M) = \inf\{k \in \mathbb{N}, \mathit{Ext}_R^k(M,R) \neq 0\}.$$

The Auslander condition on M is :

$$\forall k \in \mathbb{N}, \forall N \subset \mathit{Ext}_R^k(M,R),\ j_R(N) \geqslant k.$$

The ring R is called Auslander regular if it is noetherian, has finite global dimension and any f.g. R-module M satisfies the Auslander condition. When R is commutative, Auslander regular = regular and, in this case $j_R(M)$ is the codimension of the support of M.
Let R be Zariskian filtered. Then

- If $\mathrm{gr}_\bullet \mathrm{R}$ is Auslander regular, so is R, and $\mathrm{gld}(R) \leqslant \mathrm{gld}(\mathrm{gr}_\bullet(\mathrm{R}))$.
- For any module M on R and any "good" filtration on M we have $j_R(M) = j_{\mathrm{gr}_\bullet(\mathrm{R})}(\mathrm{gr}_\bullet(\mathrm{M}))$.

Application : If G is p-valuable, $\mathbb{Z}_p[[G]]$ is Auslander regular of global dimension $\mathrm{rank}(G) + 1$ (Venjakob, Brumer).

Application to the structure of modules

Motivated by classical Iwasawa theory ($G = \mathbb{Z}_p$), there have been attempts to understand the structure of *finitely generated torsion modules* on an Iwasawa algebra, up to *pseudo-isomorphism*.

Roughly speaking, a pseudo-isomorphism is a morphism which becomes invertible in the quotient category of $Mod_{fg}(\mathbb{Z}_p[[G]])$ by the subcategory of objects M of grade $j_{\mathbb{Z}_p[[G]]}(M) > 1$ (pseudo-null modules as defined by Venjakob).

The main result in this direction is

Theorem (Coates, Schneider and Sujatha)

Let G be p-valuable and let M be a f.g. torsion left module over $\mathbb{Z}_p[[G]]$. Then there are non-zero left ideals $L_1, \cdots, L_m$ and an injection

$$\varphi : \bigoplus_i \mathbb{Z}_p[[G]]/L_i \longrightarrow M/M_0$$

with M_0 and $\mathrm{coker}\varphi$ pseudo-null.

Topological modules over Iwasawa algebras

So far we have only considered abstract module theory over Iwasawa algebras. We now turn to topological module theory.

Definition : A *topological* $\mathbb{Z}_p[[G]]$*-module* is a linearly topologized $\mathbb{Z}_p$-module M endowed with a continous action map $\mathbb{Z}_p[[G]] \times M \longrightarrow M$. Morphisms between such objects are G-equivariant continuous $\mathbb{Z}_p$-linear maps.

Nothing interesting can be said without imposing restriction on the topology. We will be interested here in the following subcategories.

- The subctgy $\mathit{Mod}_{co}(\mathbb{Z}_p[[G]])$ of compact topological modules.
- The subctgy $\mathit{Mod}_{ad}(\mathbb{Z}_p[[G]])$ of adic modules, *i.e.* such that $M \xrightarrow{\sim} \varprojlim_n M/p^n$.

The subcategory $Mod_{co}(\mathbb{Z}_p[[G]])$

Let M be a locally compact linearly topologized $\mathbb{Z}_p$-module, and let us endow $\mathrm{End}^{\mathrm{cont}}_{\mathbb{Z}_p}(M)$ with the linear topology of compact convergence. The following data on M are equivalent :

- A continous action map $\mathbb{Z}_p[[G]] \times M \longrightarrow M$.
- A continuous map of $\mathbb{Z}_p$-algebras $\mathbb{Z}_p[[G]] \longrightarrow \mathrm{End}^{\mathrm{cont}}_{\mathbb{Z}_p}(M)_{cc}$.
- A continuous map of groups $G \longrightarrow \mathrm{End}^{\mathrm{cont}}_{\mathbb{Z}_p}(M)^{\times}_{cc}$.
- A continous action map $G \times M \longrightarrow M$.

The category $Mod_{co}(\mathbb{Z}_p[[G]])$ is abelian, and kernels and cokernels commute with the functor "forget the tgy". We denote by

$$\mathrm{Mod}^{tf}_{co}(\mathbb{Z}_p[[G]])$$

the subcategory of p-*torsion free* objects in $\mathrm{Mod}_{co}(\mathbb{Z}_p[[G]])$. It is not abelian anymore but it is quasi-abelian. This means in particular that kernels and cokernels exist. Here the cokernel of a morphism is the usal cokernel divided by the closure of the torsion submodule.

The subcategory $Mod_{ad}(\mathbb{Z}_p[[G]])$

Let M be a locally adic linearly topologized $\mathbb{Z}_p$-module, and let us endow $\mathrm{End}^{\mathrm{cont}}_{\mathbb{Z}_p}(M)$ with the linear topology of pointwise convergence. The following data on M are equivalent :

- A continous action map $\mathbb{Z}_p[[G]] \times M \longrightarrow M$.
- A continuous map of $\mathbb{Z}_p$-algebras $\mathbb{Z}_p[[G]] \longrightarrow \mathrm{End}^{\mathrm{cont}}_{\mathbb{Z}_p}(M)_{pc}$.
- A continuous map of groups $G \longrightarrow \mathrm{End}^{\mathrm{cont}}_{\mathbb{Z}_p}(M)^{\times}_{pc}$.
- A continous action map $G \times M \longrightarrow M$.

In particular the action of G is continuous if and only if it is separately continuous.

The category $Mod_{ad}(\mathbb{Z}_p[[G]])$ is only quasi-abelian, and cokernels don't commute with the functor "forget the tgy". We denote by

$$\mathrm{Mod}^{tf}_{ad}(\mathbb{Z}_p[[G]])$$

the subcategory of *torsion free* objects in $\mathrm{Mod}_{ad}(\mathbb{Z}_p[[G]])$. Again it is quasi-abelian.

Duality between adic and compact torsion free modules

For a topological torsion-free $\mathbb{Z}_p$-module M, let's put

$$M^d := \mathrm{Hom}^{cont}_{\mathbb{Z}_p}(M, \mathbb{Z}_p).$$

If M is compact, we endow M^d with the p-adic tplgy. It coincides with the compact convergence tplgy, hence is complete and M^d is an adic $\mathbb{Z}_p$-module.

If M is adic, we endow M^d with the tplgy of pointwise convergence. We then get a compact $\mathbb{Z}_p$-module.

Lemma (Shikhof)

These two constructions induce quasi-inverse anti-equivalences of categories between $\mathrm{Mod}^{tf}_{co}(\mathbb{Z}_p)$ *and* $\mathrm{Mod}^{tf}_{ad}(\mathbb{Z}_p)$*.*

Concretely any object in $\mathrm{Mod}^{tf}_{co}(\mathbb{Z}_p)$ is isomorphic to $\mathbb{Z}_p^X$ for some set X and any object in $\mathrm{Mod}^{tf}_{ad}(\mathbb{Z}_p)$ is isomorphic to the p-adic completion $\mathbb{Z}_p^{\langle X \rangle}$of $\mathbb{Z}_p^{(X)}$. Duality carries $\mathbb{Z}_p^X$ to $\mathbb{Z}_p^{\langle X \rangle}$.

Equivariant duality

Let M, N be two compact $\mathbb{Z}_p$-modules. Transposition gives a $\mathbb{Z}_p$-linear isomorphism $\mathrm{Hom}^{cont}_{\mathbb{Z}_p}(M, N) \xrightarrow{\sim} \mathrm{Hom}^{cont}_{\mathbb{Z}_p}(N^d, M^d)$.

Lemma (ST)

The above isomorphism is a homeomorphism if one endows the LHS with the tplgy of compact convergence and the RHS with the tplgy of pointwise convergence. Hence we have

$$\mathrm{Hom}^{cont}_{\mathbb{Z}_p}(M, N)_{cc} \xrightarrow{\sim} \mathrm{Hom}^{cont}_{\mathbb{Z}_p}(N^d, M^d)_{pc}.$$

Corollary (ST)

The functor $M \mapsto M^d$ induce quasi-inverse anti-equivalences of categories between $\mathrm{Mod}^{tf}_{co}(\mathbb{Z}_p[[G]])$ and $\mathrm{Mod}^{tf}_{ad}(\mathbb{Z}_p[[G]])$.

Finitely generated modules

A general result on compact rings says :

Theorem

The forgetful functor from the ctgry of f.g. topological $\mathbb{Z}_p[[G]]$-modules to the ctgry of f.g. abstract $\mathbb{Z}_p[[G]]$-modules is an equivalence of ctgries.

In particular any f.g. module over $\mathbb{Z}_p[[G]]$ carries a canonical (compact) topology making the action continuous. If $\mathbb{Z}_p[[G]]$ moreover is noetherian, e.g. if G is a p-adic Lie group, then any (left or right) ideal in $\mathbb{Z}_p[[G]]$ is closed.

Remark : duality exchanges $\mathbb{Z}_p[[G]]$ and the space $\mathcal{C}(G, \mathbb{Z}_p)$ of continous function with the sup-norm topology. So $\mathbb{Z}_p[[G]]$ is the algebra of $\mathbb{Z}_p$-valued distributions on G.

Variant

Let K be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{O}$ be its ring of integers. Then we can form the topological $\mathcal{O}$-algebra $\mathcal{O}[[G]]$. All the foregoing results on the properties of the topological ring $\mathbb{Z}_p[[G]]$ and its topological modules apply to $\mathcal{O}[[G]]$.

Banach representations

Let K be a complete non-Archimedean field. Recall that a Banach space over K is a vector space endowed with an ultrametric norm and complete w.r.t. this norm. Such spaces, with continuous K-linear maps as morphisms, form a K-linear ctgry $\mathrm{Ban}(K)$, which is quasi-abelian.

A Banach representation of a tgical group G is a Banach space with a continous action map $G \times V \longrightarrow V$. They form a ctgry $\mathrm{Ban}_G(K)$.

When K is *spherically complete* (e.g. discretely valued), Banach spaces are *barelled*, hence satisfy the so-called *Banach-Steinhaus* theorem. Therefore, if G is locally compact the following data are equivalent :

- A continuous action map $G \times V \longrightarrow V$
- A separately continuous map $G \times V \longrightarrow V$
- A continuous multiplicative map $G \longrightarrow \mathcal{L}(V, V)_s$.

Bad properties

However, the foregoing definition does not lead to a reasonable theory, even for p-adic Lie groups :

- There may exist non-trivial G-equivariant continuous maps between two non-isomorphic topologically irreducible Banach representations.
- For $G = \mathbb{Z}_p$, there are (many) infinite dimensional topologically irreducible Banach representations.

One reason for such phenomena is the lack of a K-valued Haar measure on a p-adic Lie group G, *i.e.* a left G-invariant continuous linear form on the space $\mathcal{C}(G, K)$ of continuous maps on G.

In order to get a reasonnable theory, Schneider and Teitelbaum relate Banach space representations to Iwasawa topological modules using the duality theory we have presented.

We assume from now on that K is a finite extension of $\mathbb{Q}_p$ and will describe a convenient duality for Banach space representations.

Banach spaces over K and torsion-free adic $\mathcal{O}$-modules

Let $\mathcal{O}$ be the ring of integers of K and let M be a torsion-free adic topological $\mathcal{O}$-module. Then the K-vector space $M_K := M \otimes K$ is complete for the norm $m \mapsto \sup\{|\lambda|, \lambda \in K, \lambda.m \in M\}$. This construction is functorial and we have

Lemma

The functor

$$\mathrm{Mod}_{ad}^{tf}(\mathcal{O})_{\mathbb{Q}} \longrightarrow \mathrm{Ban}(K)$$

thus obtained is an equivalence of categories.

Proof.

Let $\mathrm{Ban}(K)^{\leqslant 1}$ denote the ctgry of all Banach spaces V over K such that with $||V|| \subset |K|$, and with morphisms all norm decreasing K-linear maps. This is a $\mathcal{O}$-linear category and the obvious functor $\mathrm{Ban}(K)^{\leqslant 1}_{\mathbb{Q}} \longrightarrow \mathrm{Ban}(K)$ is an equivalence.

The functor $M \mapsto M_K$ factors through $\mathrm{Ban}(K)^{\leqslant 1}$ and the functor $V \mapsto V^{\circ}$ (unit ball) is a quasi-inverse. □

G-equivariant duality

Let G be a profinite group. We have the following equivariant version of the last lemma :

Lemma

The functor $M \mapsto M_K$ induces an equivalence of categories

$$\mathrm{Mod}_{ad}^{tf}(\mathcal{O}[[G]])_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Ban}_G(K)$$

The point is that, by compacity of G one can change the norm to an equivalent G-equivariant one.

Composing with duality between adic/compact $\mathcal{O}[[G]]$-modules :

Theorem (ST)

The functor $M \mapsto M_K^d$ induces an antiequivalence of categories $\mathrm{Mod}_{co}^{tf}(\mathcal{O}[[G]])_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Ban}_G(K)$.

This result enables Schneider and Teitelbaum to propose a convenient notion of *admissibility*.

Admissibility for Banach representations

Let G be a profinite group.
First definition : *A Banach representation of G is called admissible if it is of the form M_K^d for some finitely generated torsion-free $\mathcal{O}[[G]]$-module M.*

In other words, letting $D^c(G,K) = K[[G]] := \mathcal{O}[[G]] \otimes K$ be the algebra of distributions on G, a Banach representation V is admissible if its dual V' is a finitely generated $K[[G]]$-module.

Moreover since $\mathrm{Mod}_{fg}^{tf}(\mathcal{O}[[G]])_{\mathbb{Q}} \simeq \mathrm{Mod}_{fg}(K[[G]])$, we have an anti-equivalence of categories

$$\mathrm{Mod}_{fg}(K[[G]]) \xrightarrow{\sim} \mathrm{Ban}_G^{adm}(K).$$

In particular, if $K[[G]]$ is noetherian (*e.g.* if G is analytic) then $\mathrm{Ban}_G^{adm}(K)$ is an abelian category. Moreover kernels and cokernels commute with all forgetful functors. Thus the theory of admissible Banach representations becomes purely algebraic.

The case of p-adic Lie groups

We give here a more intrinsic formulation of the admissibility condition under the assumption that G is analytic. Let $k = \mathcal{O}/\varpi_K\mathcal{O}$ be the residue field of K.

Proposition (ST, Breuil)

A Banach representation V of G is admissible if and only if for any (resp. for one) open p-valuable subgroup H of G and any (resp. one) H-invariant bounded open $\mathcal{O}$-submodule $N \subset V$, the k-vector space $(N \otimes_{\mathcal{O}} k)^H$ is finite dimensional.

Proof : Let V, N as above and put $M := N^d \in \mathrm{Mod}_{co}^{tf}(\mathcal{O}[[H]])$, so that $V = M_K^d$. By topological Nakayama lemma for compact modules, M is f.g. over $\mathcal{O}[[H]]$ iff $\dim_k(M/R_H M) < \infty$. where R_H is the radical of $\mathcal{O}[[H]]$. Therefore the claim follows from the bijectivity of the following injection

$$\begin{aligned}(N \otimes_{\mathcal{O}} k)^H &= (\mathrm{Hom}_{\mathcal{O}}^{cont}(M, \mathcal{O}) \otimes_{\mathcal{O}} k)^H \\ &\hookrightarrow \mathrm{Hom}_{\mathcal{O}}^{cont}(M, k)^H = \mathrm{Hom}_{\mathcal{O}}(M/R_H M, k)\end{aligned}$$

Good properties of admissible Banach representations

As a consequence of the previous equivalence of categories, we get a bijection

$$\left\{ \begin{array}{l} \text{Isom. classes of} \\ \text{tplgically irreducible} \\ \text{Banach repr. of } G \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{Isom. classes} \\ \text{of simple} \\ K[[G]]-\mathrm{modules} \end{array} \right\}$$

Moreover any non-zero G-equivariant between two *admissible* tpgically irreducible Banach representations is an isomorphism, thus correcting a previously mentioned pathology

For the group $G = \mathbb{Z}_p$, since $K[[G]] \simeq K \otimes \mathcal{O}[[T]]$, we see that all topologically irreducible admissible Banach representations are *finite dimensional*, which is in accordance with the analogy with compact commutative Lie groups.

Remarks on non-compact p-adic Lie groups

A Banach representation of an arbitrary p-adic Lie group is called *admissible* if it is admissible w.r.t *any*, or equivalently *one*, compact open subgroup G_0.

Note that the dual V' of a Banach representation is a module over the distribution algebra $D^c(G,K) = \mathcal{C}(G,K)'$ and we have $D^c(G,K) = \bigoplus_{g \in G/G_0} g.K[[G_0]]$ as a $K[[G_0]]$ module.

Definition : *A Banach representation V of G is called unitary if the topology of V can be defined by a G-equivariant norm.*

Equivalently V is unitary if it contains a bounded open G-equivariant lattice N.

Sources of Banach representations

1. Spaces of continuous functions : If X is a compact set with a continuous action of G, then $\mathcal{C}(X, K)$ with the sup norm is a Banach representation of G. If X/G is finite then $\mathcal{C}(X, K)$ is admissible. Example (or variant) : principal series for reductive groups (see later).
2. Completion of representations : Start with a smooth or locally algebraic or locally analytic K-representation V of G, and assume given a G-invariant $\mathcal{O}$-submodule N which generates V over K. Define a G-invariant seminorm on V by $||v|| = \sup\{|\lambda|,\ \lambda \in K, \lambda x \in N\}$ and complete. We get (possibly zero) a unitary Banach representation $\hat{V}(N)$.
3. (φ, Γ)-modules : In their lectures, Berger and Colmez explain how to construct a Banach representation of $\mathrm{GL}_2(\mathbb{Q}_\mathrm{p})$ from a (φ, Γ)-module.
4. Completed cohomology : In his lectures, Emerton presumably will study some kind of completion of the (étale) cohomology of the modular curve with infinite p-level structure.

Principal series of reductive groups

Let G be the group of rational points of some reductive group over $\mathbb{Q}_p$ and let P be a parabolic subgroup and χ be a continuous character of P. We can form

$$\mathit{Ind}_P^G(\chi) := \{f \in \mathcal{C}(G,K),\, \forall p \in P,\, f(gp) = \chi(p)^{-1} f(g)\}.$$

Since G/P is compact, it is an admissible Banach representation of G. More precisely, choose a compact open subgroup G_0 of G such that $G = G_0.P$. E.g. if $G = \mathrm{GL_n}(\mathbb{Q}_\mathrm{p})$ and P is the Borel subgroup of upper triangular matrices, take $G_0 = \mathrm{GL_n}(\mathbb{Z}_\mathrm{p})$. Then we have

$$\mathit{Ind}_P^G(\chi)'_{|G_0} = K[[G_0]] \otimes_{K[[P \cap G_0]],\chi} K.$$

It is strongly believed that these “parabolically induced” Banach representation always have finite length.

An irreducibility conjecture

To simplify, consider the case of a *split* reductive group over $\mathbb{Q}_p$ and assume that $P = B$ is a Borel subgroup. Let T be a maximal torus in B and assume also that $\rho := \frac{1}{2} \sum_{\alpha \in \Phi(T, \mathrm{Lie}(B))} \alpha \in X^*(T)$ (this is true when G is simply connected).

Conjecture (Schneider, ICM06)

The principal series Ind$_B^G(\chi)$ is topologically irreducible unless there is a positive coroot $\alpha^\vee \in X_*(T)$ such that $(\chi\rho) \circ \alpha^\vee = (.)^m$ for some positive integer m.

Here is what is known on this conjecture :

Theorem (ST)

The conjecture is true for GL_2.

Theorem (many people)

The principal series Ind$_B^G(\chi)$ is topologically irreducible unless there is a positive coroot $\alpha^\vee \in X_*(T)$ such that $\mathbf{d}(\chi\rho \circ \alpha^\vee) = m$ for some positive integer m.

Completion of locally algebraic representations :

Let V be a smooth or locally algebraic K-representation of G, let N be a G-invariant lattice in V (not containing any K-line).

By definition the completion $\hat{V}(N)$ is admissible if and only if $N \otimes_{\mathcal{O}} k$ is admissible, as a smooth k-representation of G.

Note that $\hat{V}(N)$ indeed depends on the lattice N. More precisely, let $N' \subset N$ be another G-invariant generating $\mathcal{O}$-submodule, then the canonical G-equivariant continuous map $\hat{V}(N') \longrightarrow \hat{V}(N)$ is an isomorphism if and only if N' and N are *commensurable* which means that there is $\lambda \in K$ such that $N \subset \lambda N'$.

Question

How to parametrize classes of commensurability of lattices ?

Note that there is a distinguished comm. class. Namely that of finitely $\mathcal{O}[G]$-generated lattices.

A nice example

Let $G := \mathrm{GL}_2(\mathbb{Q}_\mathrm{p})$ and $V = V_k := \mathrm{St} \otimes \mathrm{Sym}^{k-2}(K^2) \otimes |\det|^{\frac{k-1}{2}}$, where St is the smooth Steinberg representation of G and $k > 2$.

Breuil has noticed that the completion $B(k)$ of V_k along a finitely generated lattice is not admissible.

Theorem (Breuil, Colmez)

There is a family of commensurability classes of lattices in V_k, parametrized by an element $\mathcal{L} \in K$, such that the corresponding completions $B(k, \mathcal{L})$ satisfy the following :

- *They are admissible and topologically irreducible.*
- *They are pairwise non-isomorphic*
- *The "canonical" map $B(k) \longrightarrow B(k, \mathcal{L})$ is a quotient map.*

Relevance of this construction : the $B(k, \mathcal{L})$ are the "p-adic langlands correspondent" of the semi-stable non-crystalline 2-dimensional K-representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, see Colmez lectures.